Solutions to HW 8.

1. (Exercise 7.1.3) For any set A, there is a mapping of $\mathcal{P}(A \times A)$ onto $h(A)$.

Define $f : \mathcal{P}(A \times A) \to h(A)$ by

$$f(R) = \begin{cases} \alpha & \text{if } R \text{ is a well-ordering of its field of type } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

There exists an $R \subseteq A \times A$ that is a well-ordering of its field of type α iff α is embeddable in A iff $\alpha < h(A)$. Thus, f maps onto $h(A)$.

2. (Exercise 7.1.4.) $|A| < |A| + h(A)$.

By replacing A with an equipotent set we may assume that A contains no ordinals. In this case A is disjoint from $h(A)$.

The inclusion function $\iota : A \to A \cup h(A)$ is 1-1, so $|A| \leq |A \cup h(A)| = |A| + h(A)$. If we had equality, $|A| = |A| + h(A)$, then there would exist a bijection $f : A \cup h(A) \to A$. The restriction of f to $h(A)$ would be an embedding of $h(A)$ into A. This is impossible, so $|A| \neq |A| + h(A)$, and therefore $|A| < |A| + h(A)$.

3. (Exercise 7.1.5.) $|h(A)| < |\mathcal{P}(\mathcal{P}(A \times A))|$ for all A.

It is a general fact that if $f : X \twoheadrightarrow Y$ is surjective, then

$$f^{-1} : \mathcal{P}(Y) \rightarrowtail \mathcal{P}(X) : U \mapsto f^{-1}(U)$$

is injective.

Applying this general fact to $f : \mathcal{P}(A \times A) \twoheadrightarrow h(A)$ from Problem 7.1.3 we get that $f^{-1} : \mathcal{P}(h(A)) \rightarrowtail \mathcal{P}(\mathcal{P}(A \times A))$ is 1-1. Thus $|\mathcal{P}(h(A))| \leq |\mathcal{P}(\mathcal{P}(A \times A))|$. Since by Cantor's Theorem, we have $|h(A)| < |\mathcal{P}(h(A))|$ we get $|h(A)| < |\mathcal{P}(h(A))| \leq |\mathcal{P}(\mathcal{P}(A \times A))|$.