Some Convergence Tests for Series.

Theorem 2.7.4. (Comparison Test). Assume $(a_i)_{i\in\mathbb{N}^*}$ and $(b_i)_{i\in\mathbb{N}^*}$ are sequences satisfying $0 \le a_i \le b_i$ for all $i \in \mathbb{N}^*$.

(i) If $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.
(ii) If $\sum_{i=1}^{\infty} a_i$ diverges, then $\sum_{i=1}^{\infty} b_i$ diverges.

Theorem. (Deleting First N Terms). $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=N+1}^{\infty} a_i$ both converge or both diverge.

Example 2.7.5. (Geometric Series). $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$ if $|r| < 1$. $\sum_{i=0}^{\infty} ar^i$ diverges if $|r| \ge 1$ and $a \ne 0$.

Theorem 2.7.6. (Absolute Convergence Test). If the series $\sum_{i=1}^{\infty} |a_i|$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

Exercise 2.7.9 (Ratio Test). Assume that $\sum_{i=1}^{\infty} a_i$ is a series with nonzero terms. If

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = r < 1,$$

then $\sum_{i=1}^{\infty} a_i$ converges absolutely.

Exercise 2.7.14 (Dirichlet's Test). Let $(a_i)_{i\in\mathbb{N}^*}$ be a sequence with

(i) $a_1 \ge a_2 \ge \cdots$, and
(ii) $\lim_{i\to\infty} a_i = 0$.

If $\sum_{k=1}^{\infty} b_k$ has bounded partial sums, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Theorem 2.7.7 (Alternating Series Test). Let $(a_i)_{i\in\mathbb{N}^*}$ be a sequence with

(i) $a_1 \ge a_2 \ge \cdots$, and
(ii) $\lim_{i\to\infty} a_i = 0$.

Then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

Practice! Test for Convergence!

(1) $\sum_{i=1}^{\infty} \frac{1}{n(n+1)}$

(2) $\sum_{i=1}^{\infty} \frac{1}{n^2}$

(3) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$

(4) $\sum_{n=1}^{\infty} \frac{n+1}{2^n}$

(5) $\sum_{n=1}^{\infty} \frac{n!^2}{(2n)!}$

(6) $\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n}}$