Cardinality.

(1) A set X is **finite** if there is a natural number $k \in \mathbb{N}$ and a bijection $f : k \to X$.

(2) A set X is **infinite** if it is not finite.

(3) A set X is **countably infinite** if there is bijection $f : \mathbb{N} \to X$.

(4) A set X is **countable** if it is finite or countably infinite.

(5) A set X is **uncountable** if it is not countable.

(6) $|X| \leq |Y|$ (or $|Y| \geq |X|$) means there is an injection $f : X \to Y$. We read "$|X| \leq |Y|$" as "the cardinality of X is less than or equal to the cardinality of Y". When either X or Y is a natural number, we might drop the vertical bars and write $k \leq |Y|$ or $|X| \leq k$, but these are just abbreviations for $|k| \leq |Y|$ or $|X| \leq |k|$. We also might write $k \leq \ell$ instead of $|k| \leq |\ell|$ when $k, \ell \in \mathbb{N}$.

(7) $|X| = |Y|$ means there is an bijection $f : X \to Y$. We say "the cardinality of X equal to the cardinality of Y" or X is **equipotent with** Y.

(8) $|X| < |Y|$ (or $|Y| > |X|$) means $|X| \leq |Y|$ holds but $|X| = |Y|$ fails.

The following are theorems of ZFC (= Zermelo-Fraenkel set theory with the Axiom of Choice).

Theorem 1. *A subset of a countable set is countable.*

Theorem 2. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Theorem 3. *A countable union of countable sets is countable.*

Theorem 4. *If A is an infinite alphabet and S is the set of finite strings of symbols in this alphabet, then $|A| = |S|$.*

Theorem 5. *(Cantor-Bernstein-Schröder) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Theorem 6. *(Cantor's Theorem) If X is a set, then $|X| < |\mathcal{P}(X)|$.*

Theorem 7. *If $\mathbb{F}$ is an ordered field satisfying the Nested Interval Property and $[0_\mathbb{F}, 1_\mathbb{F}]$ is the unit interval in this field, then $|[0_\mathbb{F}, 1_\mathbb{F}]| \geq |\mathcal{P}(\mathbb{N})|$.*

Theorem 8. *If $\mathbb{F}$ is an ordered field satisfying the Archimedean Property, then $|\mathbb{F}| \leq |\mathcal{P}(\mathbb{N})|$.*

Theorem 9. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| \leq |[0_\mathbb{R}, 1_\mathbb{R}]| \leq |\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$, *so* $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |[0, 1]| = |\mathbb{R}|$.