The blancmange function.

The first example of a continuous, nowhere differentiable function was given by Karl Weierstrass (1872). A simpler example was given later by Teiji Takagi (1901), which is now called the blancmange[1] function (or curve) or else the Takagi function (or curve).

To define it, let $h(x) = \inf\{|x-n| \mid n \in \mathbb{Z}\}$ be the sawtooth function of period 1. The blancmange function is

$$B(x) = h(x) + \frac{1}{2}h(2x) + \frac{1}{4}h(4x) + \cdots = \sum_{k=0}^{\infty} \frac{1}{2^k} h(2^k x).$$

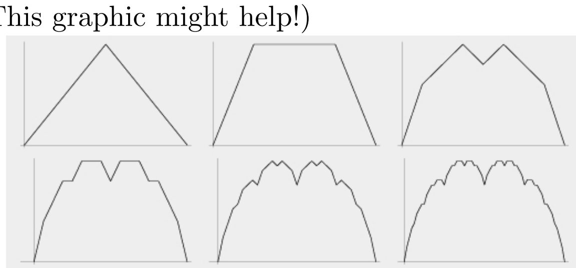
That is, $B(x) = \sum_{k=0}^{\infty} h_k(x)$ where $h_k(x) = \frac{1}{2^k} h(2^k x)$.

General theorems from Chapter 6 can be applied to prove that the blancmange function is continuous (Weierstrass M-Test and Uniform Limit Theorem). Here we argue that it is differentiable nowhere.

We will refer to *dyadic rationals*, which are rational numbers which have a representation as $\frac{m}{2^k}$ whose numerator is an integer and whose denominator is a positive integer power of 2. (For the purposes of this handout, say that $\frac{m}{2^k}$ is a *weight-k* representation of a dyadic rational.)

(1) Draw h_0, h_1, h_2 on the same coordinate system. Use different colors for different h_n's if you can. On a different coordinate system, draw the first few partial sums for $B(x)$:
$h_0(x), \quad h_0(x) + h_1(x), \quad h_0(x) + h_1(x) + h_2(x)$.

(This graphic might help!)



[1] "blancmange" refers to a white, puddinglike dessert made of sugar, cream, gelatin, and spices.

(2) Convince yourself why, if $n \geq k$, $h_n(\frac{m}{2^k}) = 0$.

(3) Convince yourself why, if $a_k := \frac{m}{2^k}$ and $b_k := \frac{m+1}{2^k}$ are consecutive dyadic rational of weight k, and $n < k$, the function $h_n(x)$ is linear on the the interval $[a_k, b_k]$, and that this linear function has slope $+1$ or -1. Moreover, this slope equals the derivative of $h_n(x)$ from the right at any point $c \in [a_k, b_k)$. (Write this slope as $h_n^+(c)$.)

(4) We wish to show that $B(x)$ is not differentiable at $x = c$ for arbitrarily chosen c. For this purpose, choose some $c \in \mathbb{R}$ which will remain fixed for the rest of this worksheet.

For this part, show that, for any weight k, it is possible to find consecutive dyadic rational a_k and b_k of weight k such that $a_k \leq c < b_k$.

(5) For a_k, c, b_k as in the last part, explain these equalities:

$$\frac{B(b_k) - B(a_k)}{b_k - a_k} = \frac{h_0(b_k) - h_0(a_k)}{b_k - a_k} + \cdots + \frac{h_{k-1}(b_k) - h_{k-1}(a_k)}{b_k - a_k} = h_0^+(c) + \cdots + h_{k-1}^+(c).$$

(6) Argue that, if $B'(c)$ existed, then the infinite series $\sum_{k=0}^{\infty} h_k^+(c)$ would have to converge to $B'(c)$.

(7) Explain why $B'(c)$ cannot exist.