ANALYSIS 1 (MATH 3001): REVIEW SHEET 2

Final Exam: Dec 18, 1:30-4pm

VI. Series. Sections 2.7, 2.9.
- (a) Definition of series.
- (b) Definition of convergent/divergent series.
- (c) Algebraic Limit Theorem for Series.
- (d) Cauchy Criterion for Series.
- (e) Absolute and conditional convergence.
- (f) Criteria for convergence/divergence of series: Comparison Test, Nth term test for Divergence, Geometric Series Theorem, telescoping series, Absolute Convergence Test, Alternating Series Test, Dirichlet's Test.
- (g) Riemann Rearrangement Theorem.

VII. Topology. Sections 3.1-3.3, 3.6.
- (a) Definitions: topology/topological space, open and closed sets, interior and closure, neighborhood, limit point, boundary point, interior point, isolated point.
- (b) Examples: discrete topology, trivial topology, cofinite topology.
- (c) Metric/metric topology, ε-ball.
- (d) Compact sets.
- (e) Heine-Borel Theorem (characterizing compact subsets of $\mathbb{R}$ in terms of limits, in terms of the metric topology, and in pure topological terms).
- (f) The connected subsets of $\mathbb{R}$ are the intervals.
- (g) A subset of $\mathbb{R}$ is compact and connected iff it is a closed bounded interval.
- (h) Cantor set. (A compact set of size $|\mathbb{R}|$ whose connected subsets are its singleton subsets.)

VIII. Continuous functions. Sections 4.1-4.5, 4.7.
- (a) Definitions (in terms of limits, in terms of the metric topology, and in pure topological terms). Continuity at a point.
- (b) (non-)Examples of Dirichlet and Thomae.
- (c) Algebraic Continuity Theorem.
- (d) (Bolzano) The continuous image of a compact set is compact.
- (e) (Bolzano) The continuous image of a connected set is connected.
- (f) Extreme Value Theorem.
- (g) Intermediate Value Theorem.
- (h) Definition of uniform continuity.
- (i) Heine-Cantor Theorem (a continuous function on a compact set is uniformly continuous).

IX. Derivatives. Sections 5.1, 5.2, 5.4, 5.5.
- (a) Definition.

(b) Differentiability implies continuity.
(c) Algebraic Differentiability Theorem.
(d) Interior Extremum Theorem.
(e) Darboux's Theorem. (Derivatives have the IVP.)
(f) Darboux's Corollary. (If f is differentiable on $[a, b]$, then f' has the IVP on $[a, b]$.)
(g) ~~If $f' > 0$ on $[a, b]$, then f is monotone increasing on $[a, b]$. If $f' < 0$ on $[a, b]$, then f is monotone decreasing on $[a, b]$.~~
(h) ~~If $f' = 0$ on $[a, b]$, then f is constant on $[a, b]$. If $f' = g'$ on $[a, b]$ and $f(c) = g(c)$ for some $c \in [a, b]$, then $f = g$ on $[a, b]$.~~
(i) The blancmange function is continuous and periodic, but nowhere differentiable.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Practice!

(1) Define the word or phrase.
  (a) Convergent series.
  (b) Conditionally convergent series.
  (c) Topology.
  (d) Interior point.
  (e) Metric.
  (f) Compact set.
  (g) Connected set.
  (h) Cantor set.
  (i) Uniformly continuous function.
  (j) Derivative of a function.

(2) State the theorem.

(a) Cauchy Criterion for series.
(b) Riemann Rearrangement Theorem.
(c) Dirichlet's Test for convergence.
(d) Heine-Borel Theorem.
(e) Heine-Cantor Theorem.
(f) Intermediate Value Theorem.
(g) Darboux's Theorem.

(3) True or False?
 (a) If an infinite series converges to 1, and it is possible to rearrange the terms to get a series that converges to 2, then it is also possible to rearrange the terms to get a series that converges to 3.
 (b) There is a function that is continuous at exactly one point.
 (c) The union of two compact sets is compact.
 (d) A closed subset of a compact set is compact.
 (e) Every compact subset of $\mathbb{R}$ is contained in a connected subset of $\mathbb{R}$.
 (f) Every connected subset of $\mathbb{R}$ is contained in a compact subset of $\mathbb{R}$.

(4) Give a counterexample to each statement.
 (a) Bounded alternating series converge.
 (b) The continuous image of a bounded open set is bounded.
 (c) The continuous image of a bounded open set is open.
 (d) Every continuous function is uniformly continuous.
 (e) The product of two uniformly continuous functions is uniformly continuous.
 (f) Periodic functions are uniformly continuous.

(5) What is being expressed by the following statement?

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(0 < |x - x_0| < \delta \to \left|\frac{x^2 - x_0^2}{x - x_0} - 1\right| < \varepsilon\right)$$

(6) Write a formal sentence expressing that f is uniformly continuous.

(7) Write a formal sentence expressing that "$f'(0) = 0$" for the function $f(x) = |x|$. Then show that the sentence is false by giving a strategy for the appropriate quantifier.