ANALYSIS 1 (MATH 3001): REVIEW SHEET

I. Discrete math review
   (a) Mathematics is intended to be well founded. (Understand the relation of primitivity with respect to definitions and theorems.)
   (b) Definitions of “function”, “natural number”, “finite”.
   (c) Reading, writing, and interpreting formal sentences.
   (d) Quantifier games for determining truth.
   (e) Axioms 1-7 of set theory. (Extensionality, Empty Set, Infinity, Pairing, Union, Comprehension, Power Set.)

II. Development of number systems
   (a) How is $\mathbb{N}$ defined? (Axiom of Infinity.)
   (b) How is $\mathbb{Z}$ constructed from $\mathbb{N}$?
   (c) How is $\mathbb{Q}$ constructed from $\mathbb{Z}$?
   (d) How is $\mathbb{R}$ constructed from $\mathbb{Q}$? (Cauchy sequences, null sequences.)

III. Review of Cardinality
   (a) Definitions of $|A| \leq |B|$, $|A| = |B|$, $|A| < |B|$, equipotence.
   (b) Definitions of finite, infinite, countably infinite, countable and uncountable.
   (c) Cantor’s Theorem, Cantor-Bernstein-Schröder Theorem, the theorem asserting that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, the theorem asserting that a subset of a countable set is countable, and the theorem asserting that a countable union of countable sets is countable.
   (d) $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

IV. Ordered fields and the Axiom of Completeness (AoC)
   (a) Axioms for ordered fields, and immediate consequences.
   (b) AoC. Suprema and infima. AoC implies its dual statement.
   (c) AoC is equivalent to the Nested Interval Property + the Archimedean Property.
   (d) The Archimedean Property is equivalent to the density of rationals in any ordered field.
   (e) Example of a non-Archimedean ordered field.
   (f) If $\mathbb{F}$ is an Archimedean ordered field and $\mathbb{K}$ is a complete ordered field, then $\mathbb{F}$ has a unique order-embedding into $\mathbb{K}$, and this embedding preserves the field operations.
   (g) An Archimedean ordered field has cardinality at most $|\mathcal{P}(\mathbb{N})|$. An ordered field with the Nested Interval Property has cardinality at least $|\mathcal{P}(\mathbb{N})|$. A complete ordered field has cardinality exactly $|\mathcal{P}(\mathbb{N})|$.
   (h) There is a unique complete ordered field up to isomorphism.

V. Sequences.
  (a) Definition of metric: a positive definite, symmetric 2-variable function satisfying the triangle inequality. Standard metric for the real numbers is $d(x, y) = |x - y|$.
  (b) Definition of sequence. Definition of subsequence.
  (c) Definition of convergent/divergent sequence.
  (d) Limit Theorems:
    (i) Algebraic Limit Theorem.
    (ii) Order Limit Theorem.
    (iii) Monotone Convergence Theorem.
    (iv) Bolzano-Weierstrass Theorem.
    (v) Cauchy Criterion.
  (d) Criteria for divergence.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Common types of test problems.

(i) Tell who did what. (Who wrote the Elements? [Euclid] Who published the formula for the roots of a cubic polynomial? [Cardano] ETC.)
(ii) Give the definition of a word or phrase.
(iii) Give an example.
(iv) True or false.
(v) Calculate a value.
(vi) Easy proof or explanation.

Sample Practice Problems.

(1) Define the terms.
  (i) Archimedean (of an ordered field). (You may assume that the definition of ordered field is known.)
  (ii) Divergent sequence.

(iii) Equipotence.

(iv) Supremum.

(2) True or False? Show/Explain.

(i) $(0, 1, 2, \ldots)$ diverges.

(ii) Any complete ordered field is Archimedean.

(iii) Every null sequence converges.

(iv) $|\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{R})|$.

(3) State the theorem.

(i) Cantor-Bernstein-Schröder Theorem.

(ii) Cantor's Theorem.

(iii) Monotone Convergence Theorem.

(4) What is being expressed by the sentence

$$(\forall x)(\forall y)((x < y) \rightarrow (\exists z)((x < z) \wedge (z < y)))?$$

Give an example of an ordered structure that satisfies this sentence and another ordered structure that does not satisfy it.

(5) Write a formal sentence that expresses that $(1, 1, 1, \ldots)$ diverges.