Solutions to HW 9.

1. (Exercise 4.3.6(a)(b)(c).) Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x)+g(x)$ are continuous at 0.

Let f be the Dirichlet function, namely

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else.} \end{cases}$$

Let $g(x) = 1 - f(x)$, that is

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{else.} \end{cases}$$

Neither f nor g is continuous anywhere, but $f(x)+g(x) = 1$ is a continuous (constant) function, and $f(x)g(x) = 0$ is a continuous (constant) function.

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)+g(x)$ is continuous at 0.

This can't happen. If $f(x)+g(x)$ and $f(x)$ are continuous at 0, then $(f(x)+g(x)) - f(x) = g(x)$ is continuous at 0.

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

Let $f(x) = 0$ be the constant zero function and let g be any discontinuous function (like the Dirichlet function). Then $f(x)g(x) = 0$ is constant zero, hence continuous.

2. and 3. The goal is to show that a nonempty subset $C \subseteq \mathbb{R}$ is closed iff there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $C = g^{-1}(0)$.

For 2: Show the IF part. (Hint: explain why the inverse image of a closed set is closed.)

For 3: Show the ONLY IF part. (Hint: you may cite parts of Exercise 4.3.12 if needed.)

Following the hint, we prove first the **Claim** that if $g : \mathbb{R} \to \mathbb{R}$ is continuous, then the inverse image $g^{-1}(C)$ of a closed set C is closed.

Proof of Claim. Assume that g is continuous and that C is closed. We must show that $g^{-1}(C)$ is closed.

Since C is closed, $\mathbb{R} \setminus C$ is open. Since g is continuous, $g^{-1}(\mathbb{R} \setminus C)$ is open. Now an element x belongs to this open set iff

$$\begin{aligned} x \in g^{-1}(\mathbb{R} \setminus C) \quad & \text{iff} \quad g(x) \in \mathbb{R} \setminus C \\ & \text{iff} \quad g(x) \notin C \\ & \text{iff} \quad x \notin g^{-1}(C) \\ & \text{iff} \quad x \in \mathbb{R} \setminus g^{-1}(C). \end{aligned}$$

Hence the open set $g^{-1}(\mathbb{R} \setminus C)$ equals $\mathbb{R} \setminus g^{-1}(C)$. But if this latter set is open, then its complement $g^{-1}(C)$ must be closed. End of Proof of Claim.

For Problem 2: As just explained, if g is continuous, then the inverse image $g^{-1}(C)$ of any closed set C is closed. Since singleton sets in $\mathbb{R}$ are closed, it follows that $g^{-1}(0)$ is closed.

For Problem 3: assume that $C \subseteq \mathbb{R}$ is closed. It is shown in Exercise 4.3.12 that

$$g(x) = \inf\{|x - c| \mid c \in C\}$$

is a continuous function that does not vanish off of C. (In fact, $d \notin C$ implies that $g(d) > 0$.) It is easy to see that g does vanish on C, that is $g(b) = 0$ for any $b \in C$. To see why, note that if $b \in C$, then $g(b)$ is the infimum of the set $\{|b - c| \mid c \in C\}$. But the elements of this set are nonnegative, and one of them is zero, so the infimum is zero.

This shows that $g^{-1}(0) = C$. Altogether, we get that an arbitrarily chosen closed set C can be represented as the inverse image of $\{0\}$ under some continuous function g.