Solutions to HW 8.

1. (Exercise 3.2.3.) Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ε–neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$.

Neither open nor closed. There is no open ball $B(1, \varepsilon)$, $\varepsilon > 0$, that is contained entirely in $\mathbb{Q}$. Hence 1 is not an interior point of $\mathbb{Q}$. Hence $\mathbb{Q}$ cannot be open in $\mathbb{R}$. Also, $\sqrt{2} \notin \mathbb{Q}$ is a limit point of $\mathbb{Q}$ that is not in $\mathbb{Q}$. Hence $\mathbb{Q}$ is not closed.

(b) $\mathbb{N}$.

Closed. But not open, since $1 \in \mathbb{N}$ is not an interior point of $\mathbb{N}$.

(c) $X := \{x \in \mathbb{R} \mid x \neq 0\}$.

Open. Not closed, since $0 \notin X$ is a limit point not in X.

(d) $Y := \{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \mid n \in \mathbb{N}^*\}$.

Neither open nor closed. $1 \in \mathbb{Y}$ is not an interior point, so not open. The infinite sum $\sum_{k=1}^{\infty} 1/k^2$ is a limit point of Y not in Y. The reason this value is not in Y is that the partial sums of $\sum_{k=1}^{\infty} 1/k^2$ are positive and strictly increasing, so the limit is greater than any partial sum.

(e) $Z := \{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \mid n \in \mathbb{N}^*\}$.

Closed, since Z has no limit points. (Any open ball $B(p, \varepsilon)$ contains only finitely many points of Z, since the Harmonic series diverges to $+\infty$.) But not open, since $1 \in \mathbb{Z}$ is not an interior point.

2. (Exercise 3.3.4.) Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$.

Both. ($K \cap F$ is both closed and bounded, so closed and compact.)

(b) $\overline{F^c \cup K^c}$.

Closed, but may not be compact. (If $K = F = \emptyset$, then $\overline{F^c \cup K^c} = \mathbb{R}$.)

(c) $K \setminus F = \{x \in K \mid x \notin F\}$.

Possibly neither. For example, let $K = [0, 1]$ and $F = \{0\}$. Then $K \setminus F = (0, 1]$, which is neither closed nor compact.

(d) $\overline{K \cap F^c}$.

Both. First, $\overline{K \cap F^c}$ is closed by construction. (The overbar forces it to be closed.) Second, it is contained in K, so it is bounded, and this makes it compact.

3. (Exercise 3.3.6.) Verify that the following three statements are true if every blank is filled in with the word "finite." Which are true if every blank is filled in with the word "compact"? Which are true if every blank is filled in with the word "closed"?

I. Explain why the statements are true for *finite*, give examples to show that they are false for *closed*, and then state (without proof or counterexample) whether they are true or false for *compact*.

II. I think the author meant all sets to be nonempty in this problem, especially part (a).

I explain why all parts are true for "nonempty+finite", I assert without proof that they are all true for "nonempty+compact", and I give an example to show they are all false for "closed".

(a) We can prove that a nonempty finite set $A = \{a_1 \dots, a_k\}$ has a maximum by induction. If $A = \{a_1\}$, then $\max\{A\} = a_1$. Now suppose $A = \{a_1, \dots, a_k, a_{k+1}\}$ is given. By induction, $\{a_1, \dots, a_k\}$ has a maximum element, say a_i. Then $\max\{a_i, a_{k+1}\}$ is the maximum of A. (If A is empty, then it has no maximum element because it has no element.)

Assertion: Every nonempty compact set A has a maximum. (If A is empty, then A is compact, but it has no maximum element.)

Not true for closed. Example: $A = \mathbb{R}$ has no maximum element.

(b) If $|A| = m$ and $|B| = n$, then $|A + B| \leq m \cdot n$, so if A and B are finite, then $A + B = \{a + b \mid a \in A, b \in B\}$ is also finite.

Assertion: If A and B are compact, then $A + B = \{a + b \mid a \in A, b \in B\}$ is also compact.

Not true for closed. Example: $A = \mathbb{Z}$, $B = \{2 + \frac{1}{2}, 3 + \frac{1}{3}, 4 + \frac{1}{4}, \dots\}$. $A + B$ contains no integers, but does contain $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Thus $0 \in \overline{(A + B)} \setminus (A + B)$.

(c) If $\{A_n \mid n \in \mathbb{N}\}$ are finite, and every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well. To see this, consider the sets

$$A_1 \supseteq A_1 \cap A_2 \supseteq A_1 \cap A_2 \cap A_3 \supseteq \dots.$$

These are finite intersections, so they are all nonempty. This is a descending chain of subsets of the finite set A_1. A_1 has only finitely many subsets, so for some N we have

$$\bigcap_{i=1}^{N} A_i = \bigcap_{i=1}^{N+1} A_i = \dots = B \neq \emptyset.$$

But then $\bigcap_{i=1}^{\infty} A_i = B \neq \emptyset$ as well.

Assertion: If $\{A_n \mid n \in \mathbb{N}\}$ is a collection of compact sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

Not true for closed. Example: Let $A_n = [n, \infty)$.