Solutions to HW 7.

1. (Exercise 2.7.14(a).) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

The differences: Dirichlet's Test assumes that $\lim y_k = 0$, while Abel's Test does not. Abel's Test assumes that $\sum x_k$ converges, while Dirichlet's Test assumes only that the sequence of partial sums is bounded.

To prove Dirichlet's Test, we will use the summation by parts formula:

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}). \tag{2.7.12}$$

Here $s_k = \sum_{n=1}^{k} x_n$ is the kth partial sum of the series $\sum_{n=1}^{\infty} x_n$. To show that the left hand side of (2.7.12) has a limit, it suffices to show that the right hand side has a limit. For this, it will suffice to show that the series on the right hand side is absolutely convergent. Let M be a bound such that $|s_k| \leq M$ for all k. Now

$$\sum_{k=1}^{n} |s_k(y_k - y_{k+1})| \leq \sum_{k=1}^{n} M(y_k - y_{k+1}) = My_1 - My_{n+1} \leq My_1.$$

Since $\sum_{k=1}^{\infty} |s_k(y_k - y_{k+1})|$ has bounded partial sums, $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ is absolutely convergent, hence $\sum_{k=1}^{\infty} x_k y_k$ is convergent.

2. Let θ be an arbitrary angle. Show that the partial sums of $\sum_{k=1}^{\infty} \sin(k\theta)$ are bounded. Hint: You may want to first prove and use the equality

$$2\sin(k\theta)\sin(\theta/2) = \cos((k-1/2)\theta) - \cos((k+1/2)\theta).$$

The angle addition formula implies that

$$\begin{aligned} \cos((k-1/2)\theta) &= \cos(k\theta)\cos(\theta/2) + \sin(k\theta)\sin(\theta/2) \\ \cos((k+1/2)\theta) &= \cos(k\theta)\cos(\theta/2) - \sin(k\theta)\sin(\theta/2). \end{aligned}$$

Subtracting yields

$$\cos((k-1/2)\theta) - \cos((k+1/2)\theta) = 2\sin(k\theta)\sin(\theta/2).$$

If $\sin(\theta/2) \neq 0$, then we can divide to obtain that

$$\sin(k\theta) = \frac{\cos((k-1/2)\theta) - \cos((k+1/2)\theta)}{2\sin(\theta/2)}.$$

This means that the nth partial sum of $\sum \sin(k\theta)$ is

$$\begin{aligned}
\sum_{k=1}^{n} \sin(k\theta) &= \sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) \\
&= \tfrac{\cos((1/2)\theta) - \cos((3/2)\theta)}{2\sin(\theta/2)} + \tfrac{\cos((3/2)\theta) - \cos((5/2)\theta)}{2\sin(\theta/2)} + \cdots + \tfrac{\cos((n-1/2)\theta) - \cos((n+1/2)\theta)}{2\sin(\theta/2)} \\
&= \tfrac{\cos((1/2)\theta) - \cos((n+1/2)\theta)}{2\sin(\theta/2)} \\
&\le \tfrac{2}{2\sin(\theta/2)} = \tfrac{1}{\sin(\theta/2)}.
\end{aligned}$$

This upper bound on partial sums is independent of n, so the partial sums are bounded.

The preceding argument required the assumption that $\sin(\theta/2) \neq 0$. If instead $\sin(\theta/2) = 0$, then θ must be an even integer multiple of π, so $k\theta$ is an even integer multiple of π for all k, so $\sin(k\theta) = 0$ for all k. In this case the partial sums of $\sum \sin(k\theta)$ are still bounded, since they are all zero.

3. (a) Let θ be an arbitrary angle. Show that $\sum_{k=1}^{\infty} \sin(k\theta)/k$ converges.

   Here we just cite Dirichlet's Test with $y_k = 1/k$ and $x_k = \sin(k\theta)$.

   (b) Show that $\sum_{k=1}^{\infty} \sin(k\theta)/\sqrt{k}$ converges.

   Here we just cite Dirichlet's Test with $y_k = 1/\sqrt{k}$ and $x_k = \sin(k\theta)$.