Solutions to HW 6.

1. (Exercise 2.6.2 (a) (c).) Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

Let $x_n = (-1)^n/(n+2)$. Then $(x_n)_{n\in\mathbb{N}} = (1/2, -1/3, 1/4, -1/5, 1/6, \ldots)$ converges to zero, so it is Cauchy. It is not monotone increasing since $1/2 > -1/3$, and it is not monotone decreasing since $-1/3 < 1/4$. Hence it is not monotone.

(c) A divergent monotone sequence with a Cauchy subsequence.

Impossible. We shall argue that a monotone sequence with a Cauchy subsequence must be convergent.

Assume that (a_i) is a monotone *increasing* sequence with a Cauchy subsequence (a_{i_j}). Then $(\exists L)(\lim a_{i_j} = L)$, according to the Cauchy Criterion. By the proof of the Monotone Convergence Theorem, L is the supremum of the set of terms in the subsequence. We shall argue that L is an upper bound for the terms of the original sequence. Then the MCT implies that the original sequence converges.

Choose any term a_j from the original sequence. Then since $j \leq i_j$, and since the sequence is assumed to be monotone increasing, we have $a_j \leq a_{i_j}$. We also have $a_{i_j} \leq L$, since L is the supremum of the terms in the subsequence. By transitivity, $a_j \leq L$, so L is an upper bound for the terms of the original sequence.

2. (Exercise 2.6.4 (c)) Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is(= MUST BE) a Cauchy sequence, justifying each conclusion.

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

$([[a_n]])_{n\in\mathbb{N}}$ need not be a Cauchy sequence.

The sequence $(a_n)_{n\in\mathbb{N}} = (1/2, -1/3, 1/4, -1/5, 1/6, \ldots)$, with $a_n = (-1)^n/(n+2)$, is a Cauchy sequence converging to zero. But $([[a_n]])_{n\in\mathbb{N}} = (0, -1, 0, -1, 0, -1, 0, \ldots)$ is not Cauchy.

3. (Exercise 2.7.4 (a) (d)) Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

It's possible.

By the Nth term test for divergence, both $\sum x_n$ and $\sum y_n$ diverge if $(x_n)_{n\in\mathbb{N}} = (0, 1, 0, 1, 0, 1, \ldots)$ and $y_n = 1 - x_n$ (so $(y_n)_{n\in\mathbb{N}} = (1, 0, 1, 0, 1, 0, \ldots)$). But $x_n y_n = 0$ for all n, so $\sum x_n y_n = 0 + 0 + 0 + \cdots$ converges to zero.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.

It's possible.

(Here I will index the sequence with $\mathbb{N}^*$, which starts at 1.)

Let $(x_n)_{n\in\mathbb{N}^*} = (0, 1/2, 0, 1/4, 0, 1/6, 0, 1/8, \ldots)$. This sequence has its even terms equal to the even terms of the Harmonic Series and its odd terms equal to zero.

The sum $\sum(-1)^n x_n = -0+1/2-0+1/4-0+1/6-0+\cdots = (1/2)(1+1/2+1/3+\cdots)$ diverges, since it is (term-by-term) half the Harmonic Series.