**Solutions to HW 5.**

1. (Exercise 2.4.2.)

(a) Consider the recursively defined sequence $y_1 = 1$, $y_{n+1} = 3 - y_n$, and set $y = \lim y_n$. Because $(y_n)$ and $(y_{n+1})$ have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for $y$, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

$(y_n)$ is a sequence of integers, so it can't have a non-integer limit. The flaw in the argument is the assumption that $(y_n)$ and $(y_{n+1})$ converge.

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Yes. This sequence,

$$(y_n) = (1, 2, 5/2, 13/5, 34/13, \ldots) = (1,\ 2,\ 2.5,\ 2.6,\ 2.615, \ldots)$$

is a monotone bounded sequence, so it must converge. If $\lim y_n = L$, then applying the Algebraic Limit Theorem to $y_{n+1} = 3 - \frac{1}{y_n}$ we get $L = 3 - 1/L$ or $L^2 - 3L + 1 = 0$. This implies that $L = \frac{3 \pm \sqrt{5}}{2}$. Since all of the terms of the sequence are at least 1, $L \geq 1$, so $L \neq \frac{3-\sqrt{5}}{2} \approx .382$. This implies that $L = \frac{3+\sqrt{5}}{2} = 1 + \phi \approx 2.618$.

To show that the sequence is monotone and bounded, it suffices to prove by induction that $1 \leq y_n < y_{n+1} \leq 3$. The base case is easy, and the inductive step can be argued as follows:

$$\begin{array}{rcccccccll}
 & 1 & \leq & y_n & < & y_{n+1} & \leq & 3, & & \text{Assumption} \\
 & -1 & \leq & -1/y_n & < & -1/y_{n+1} & \leq & -1/3, & & \text{Negative inverse.} \\
 & 2 & \leq & 3 - 1/y_n & < & 3 - 1/y_{n+1} & \leq & 3 - 1/3, & & \text{Add 3} \\
 & \| & & \| & & \| & & \| & & \\
1 \leq & 2 & \leq & y_{n+1} & < & y_{n+2} & \leq & 3 - 1/3 & \leq 3 & \text{Conclusion}
\end{array}$$

2. (Exercise 2.4.3.)

(a) Show that

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

converges and find the limit.

Yes. Follow the October 4 lecture notes. The limit is $L = 2$.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Yes. Use the recursion $a_0 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. The sequence is monotone increasing and bounded. (Show this by establishing that $1 \le a_n < a_{n+1} \le 2$ for all $n$.) The limit satisfies $L^2 = 2L$, so $L = 0$ or $L = 2$. $L = 0$ is impossible, so $L = 2$.

3. (Exercise 2.5.1 (a), (b), (c)) Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

Can't happen. If $(a_i)$ is a sequence and $\left(a_{i_j}\right)$ is a bounded subsequence, then by Theorem 2.5.5 the sequence $\left(a_{i_j}\right)$ has a convergent subsequence, say $\left(a_{i_{j_k}}\right)$. But this is a convergent subsequence of the original sequence.

(b) A sequence that does not contain 0 or 1 as a term but contains subse- quences converging to each of these values.

$$\left(\frac{1}{2},\ 1+\frac{1}{2},\ \frac{1}{3},\ 1+\frac{1}{3},\ \frac{1}{4},\ 1+\frac{1}{4},\ \ldots\right) = \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{5}{4}, \ldots\right).$$

Even terms converge to 0, odd terms converge to 1. None of the terms are integers.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

It suffices to create a sequence where each number $1/n$ appears infinitely often in the sequence. Then each number $1/n$ will be the limit of a constant subsequence.

To construct such a sequence, define every other term to be 1. Define "half" of the remaining terms to be 1/2. Define "half" of the remaining terms to be 1/3. ETC. This yields

$$\left(1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{2}, 1, \frac{1}{5}, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{2}, 1, \frac{1}{4}, 1, \ldots\right),$$

which works.