Solutions to HW 3.

1. (Exercise 1.5.2.) Find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable: Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ and construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{i=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{i=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

   The error is the assumption that NIP holds for $\mathbb{Q}$.

2. (Exercise 1.5.8.) Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countably infinite.

   Let $B_k = B \cap \left[\frac{1}{k}, \infty\right)$. Any sum of more than $2k$ members of B_k will exceed $2k \cdot \frac{1}{k} = 2$, so it must be that $|B_k| \leq 2k$. By the Archimedean property of $\mathbb{R}$, $B = \bigcup_{i=1}^{\infty} B_k$. Hence B is a countable union of countable sets, making B countable. (Countable = finite or countably infinite.)

3. Show that if $S \subseteq [0,1]$ is uncountable, then there is a real number $r \in [0,1]$ such that both $[0,r] \cap S$ and $[r,1] \cap S$ are uncountable.

   Define
$$A = \{r \in [0,1] \mid [0,r] \cap S \text{ is countable}\},$$
and
$$B = \{r \in [0,1] \mid [r,1] \cap S \text{ is countable}\}.$$
The set A is nonempty ($0 \in A$) and bounded above by 1, so $\sup(A)$ exists. Similarly $\inf(B)$ exists.

   Claim 1. $\sup(A) < \inf(B)$.

   Proof of Claim. If the claim is not true, then $\inf(B) \leq \sup(A)$, so there is a real number t such that $\inf(B) \leq t \leq \sup(A)$. For each n we have that $t - (1/n) < \sup(A)$, so by Lemma 1.3.8 there is a number $a_n \in A \cap (t - (1/n), 1]$. Similarly, there is a number $b_n \in B \cap [0, t + (1/n))$. This implies that $S \cap [0, t - (1/n)]$ and $S \cap [t + (1/n), 1]$ are both countable. Since a countable union of countable sets is countable,
$$\begin{aligned} S &= S \cap [0,1] \\ &= S \cap \left(\bigcup_{n=1}^{\infty}[0, t - 1/n] \cup \bigcup_{n=1}^{\infty}[t + (1/n), 1] \cup \{t\}\right) \\ &= \left(\bigcup_{n=1}^{\infty} S \cap [0, t - (1/n)]\right) \cup \left(\bigcup_{n=1}^{\infty} S \cap [t + (1/n), 1]\right) \cup (S \cap \{t\}) \end{aligned}$$
is countable. This contradicts the uncountability of S. □

   To complete the solution, the Claim shows that $\sup(A) < \inf(B)$, so there is some real number r such that $0 \leq \sup(A) < r < \inf(B) \leq 1$. For this r we have that $S \cap [0,r]$ uncountable, else $r \in A$ and we get the contradiction that $\sup(A) < r$ yet $r \in A$. In a similar way, using B, we get that $S \cap [r,1]$ is uncountable.