Solutions to HW 2.

1. (Exercise 1.3.2.) Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

$B = \{0\}$. ($\inf B = 0 = \sup B$.)

(b) A finite set that contains its infimum but not its supremum.

The request is impossible. Any set that has an infimum is nonempty, and any finite nonempty set must contain its supremum. Thus, either a finite set is empty, in which case it has no infimum or supremum, or it is not, in which case it contains both its infimum and supremum.

To see that a finite nonempty set must contain its infimum, we argue the contrapositive: if a nonempty set does not contain its infimum, then it is infinite.

Let F be a nonempty set that does not contain its infimum. Choose any $a_0 \in F$; it cannot be the infimum. Therefore there is some $a_1 \in F$ such that $a_0 > a_1$. Now a_1 is not the infimum, so the same argument shows that there is some $a_2 \in F$ such that $a_0 > a_1 > a_2$. This process can be continued indefinitely to produce infinitely many elements of F: $a_0 > a_1 > a_2 > \cdots$.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

$B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Here $\sup(B) = 1 \in B$, while $\inf(B) = 0 \notin B$.

2. (Exercise 1.3.9.)

(a) If $\sup(A) < \sup(B)$, show that there exists an element $b \in B$ that is an upper bound for A.

Suppose that $\sup(A) = s < t = \sup(B)$. Choose $\varepsilon = t - s > 0$. By Lemma 1.3.8 there is an element $b \in B$ such that $b > t - \varepsilon = t - (t - s) = s$. Thus, $b \in B$ and $\sup(A) = s < b$, as desired.

(b) Give an example to show that this is not always the case if we only assume $\sup(A) \leq \sup(B)$.

If $A = B = (0, 1)$, then $\sup(A) = 1 = \sup(B)$, so $\sup(A) \leq \sup(B)$. But there is no $b \in B$ that is an upper bound for A, since $b \in B$ implies $b < 1$, but all upper bounds of A are $\geq \sup(A) = 1$.

3. Does the non-Archimedean field $\mathbb{R}(t)$ satisfy the Nested Interval Property? Explain.

No. To see this, let's argue that the nested intersection $\bigcap_{n=1}^{\infty}[n, t/n]$ is empty.

This is a proof by contradiction. Assume that the rational function $P(t)/Q(t)$ of $\mathbb{R}(t)$ belongs to the intersection. Here $P(t) = p_k t^k + \cdots + p_1 + p_0$ and $Q(t) = q_\ell t^\ell + \cdots + q_1 + q_0$ are polynomials with real coefficients, both must be nonzero, and both may be assumed to have positive leading coefficients (since this is true of every element of the first interval, $[1, t]$). Here the 'leading coefficient' of P is $p_k > 0$ and the 'leading coefficient' of Q is $q_\ell > 0$. Also, the 'degree' of P is $\deg(P) = k$, while the 'degree' of Q is $\deg(Q) = \ell$.

Lemma 1. *In the ordered field $\mathbb{R}(t)$, if $A(t) = a_k t^k + \cdots + a_1 + a_0$ and $B(t) = b_\ell t^\ell + \cdots + b_1 + b_0$ both have positive leading coefficients (i.e., $a_k, b_\ell > 0$), and $0 < A - nB$ for all $n \in \mathbb{N}$, then $\deg(A) > \deg(B)$.*

Proof. There are cases to consider:

(a) ($\deg(A) < \deg(B)$) The leading coefficient of $A - nB$ is $-nb_\ell < 0$, a contradiction to $0 < A - nB$.

(b) ($\deg(A) = \deg(B)$) Then $k = \ell$ and the leading coefficient of $A - nB$ is $a_k - nb_k$. If this is positive for all n, then $a_k > nb_k$ for all n, or $a_k/b_k > n$ for all n. But this contradicts the Archimedean property of $\mathbb{R}$: the real number a_k/b_k would be larger than any natural number.

(c) ($\deg(A) > \deg(B)$) This is the only remaining case, so it must hold.

Overall, our conclusion is that $\deg(A) > \deg(B)$. □

We have assumed that P/Q belongs to the intersection $\bigcap_{n=1}^{\infty}[n, t/n]$, so $n < P/Q < t/n$ for all n. The left hand inequalities, $n < P/Q$ for all n, taken together, are equivalent to the statement that $0 < P - nQ$ holds for all n. By the lemma, $\deg(P) > \deg(Q)$. The right hand inequalities, $P/Q < t/n$ for all n, are equivalent to the statement that $0 < tQ - nP$ holds for all n. By the lemma, $\deg(tQ) > \deg(P)$.

Now we have $\deg(Q) < \deg(P) < \deg(tQ) = 1 + \deg(Q)$. That is, the positive integer $\deg(P)$ lies strictly between the consecutive positive integers $\deg(Q)$ and $1 + \deg(Q)$, which is impossible. This is the contradiction that completes the proof.

{Comments on this solution: How does one know (or decide) to include a lemma like the one above? Answer: If P/Q belongs to $[n, t/n]$ for all n, then $n < P/Q$ for all n and $P/Q < t/n$ for all n. One extracts conclusions from these assumptions by the same arguments. Typically you don't realize this until you have written both arguments out. But then, rather than submit the same argument twice, it is better (= shorter and clearer) to write the argument once as a lemma and refer to the lemma twice.}