Solutions to HW 1.

1. (Exercise 1.2.4.) Produce an infinite collection of sets $A_1, A_2, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

   In this problem, $\mathbb{N} = \{1, 2, \ldots\}$.
   Let A_i be the set of all natural numbers divisible by 2^{i-1} but not divisible by 2^i. (So, A_1 = set of odd natural numbers, $A_2 = 2 \cdot A_1$, $A_3 = 4 \cdot A_1$, ETC.) By unique factorization, $\{A_1, A_2, \ldots\}$ is a partition of $\mathbb{N}$ into infinitely many infinite classes.

2. (Exercise 1.2.7.)
   (b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

   $A = (0, \infty) \subseteq \mathbb{R}$, $B = (-\infty, 0) \subseteq \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$. Then $f(A \cap B) = f(\emptyset) = \emptyset$, but $1 \in f(A) \cap f(B)$ since $f(1) = 1 = f(-1)$ and $1 \in A$, $-1 \in B$.

   (c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

   We must show that $(\forall y)((y \in g(A \cap B)) \to (y \in g(A) \cap g(B)))$.
   Choose any $y \in g(A \cap B)$. There exist $x \in A \cap B$ such that $g(x) = y$. Since $x \in A$ we get $y \in g(A)$, and since $x \in B$ we get $y \in g(B)$. Hence $y \in g(A) \cap g(B)$. This shows that $g(A \cap B) \subseteq g(A) \cap g(B)$.

3. (Exercise 1.2.8.) ($\mathbb{N} = \{0, 1, 2, \ldots\}$.) Give an example of each or state that the request is impossible:
   (a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

   $f(n) = n + 1$

   (b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

   $$f(n) = \begin{cases} n - 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \qquad \text{(Note: } f(1) = 0 = f(0).\text{)}$$

   (c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

   If $\mathbb{N} = \{0, 1, 2, \ldots\}$, then we can take

   $$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$