Solutions to HW 10.

1. (Exercise 4.5.2.) Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

Continuous function $f(x) = \sin(x)$, open interval $= (0, 2\pi)$, range $= f((0, 2\pi)) = [-1, 1]$.

(b) A continuous function defined on a closed interval with range equal to an open interval.

This can't happen if the closed interval is bounded. For then, the closed interval would be compact and connected, hence the continuous image would be compact and connected, hence the image would be a closed bounded interval, not an open interval.

But, if you allow "closed interval" to include unbounded closed intervals, like $[0, \infty)$, then it is possible. Let the continuous function be $f(x) = x \sin(x)$, let the closed interval be $[0, \infty)$. Then $f([0, \infty)) = (-\infty, \infty)$, which is an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

Let the continuous function be $f(x) = |x|$, let the open interval be $(-\infty, \infty)$. Then $f((-\infty, \infty)) = [0, \infty)$, which is an unbounded closed set different from $\mathbb{R}$.

(d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

This can't happen. The continuous image of a connected set is connected, and $\mathbb{R}$ is connected while $\mathbb{Q}$ is not.

2. (Exercise 4.5.3.) A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

We must give a winning strategy for $\exists$ in the game

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(|x - c| < \delta \to |f(x) - f(c)| < \varepsilon)$$

for $c \in [a, b]$. We argue this for $a < c < b$ only, although the argument can be modified for $a = c$ or $c = b$.

- Let $\forall$ choose $\varepsilon > 0$.

- If $f(a) = f(c)$, then f is constant on $[a, c]$ since f is monotone increasing. If $f(a) < f(c)$, then choose u_1 so that it is between $f(a)$ and $f(c)$ and also between $f(c) - \varepsilon$ and $f(c)$. In either case, by the constancy of f on $[a, c]$ or by the IVP, there exists some $d_1 < c$ such that $f(d_1) = u_1$. Similarly, we can choose some u_2 so that it is between $f(c)$ and $f(b)$ and also between $f(c)$ and $f(c) + \varepsilon$. As before, we can find some $d_2 > c$ such that $f(d_2) = u_u$. Now let $\exists$ choose $\delta > 0$ so that $(c - \delta, c + \delta) \subseteq (d_1, d_2)$.

  By these choices, and the monotonicity of f,
  $$f(c) - \varepsilon < f(d_1) \leq f((c - \delta, c + \delta)) \leq f(d_2) < f(c) + \varepsilon.$$
- Since $f((c - \delta, c + \delta)) \subseteq (f(c) - \varepsilon, f(c) + \varepsilon)$, whatever $\forall$ chooses for x we will satisfy the condition "$|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$".

3. (Exercise 4.5.7.) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

If $f(0) = 0$, then 0 is a fixed point of f, and we are done (we found a fixed point). If $f(1) = 1$, then 1 is a fixed point of f, and we are done.

If we are not yet done, then $f(0) \in (0, 1]$, so $f(0) > 0$, and $f(1) \in [0, 1)$, so $f(1) < 1$. Hence the continuous function $g(x) = x - f(x)$ satisfies $g(0) < 0 < g(1)$. By the Intermediate Value Theorem, there is some $c \in (0, 1)$ such that $0 = g(c) = c - f(c)$, hence $f(c) = c$. In this case, c is a fixed point of f and we are done.