Practice with fields and ordered fields!

Explain why the following are true.

(1) Any field satisfies $x \cdot 0 = 0$ for any x.

(2) Any field satisfies $-(-x) = x$ for any x. (Hint: Start by explaining why $(-x) + (-(-x)) = 0 = (-x) + x$. Then explain why the leftmost $-x$ can be deleted from the left and right sides.)

(3) Any field satisfies $(-x)^2 = x^2$ for any x.

(4) In an ordered field,

(a) If $0 < x$, then $0 < x^2$.

(b) If $x < 0$, then $0 < -x$.

(c) If $x \neq 0$, then $0 < x^2$.

(d) Sums of nonzero squares are positive. (That is, $0 < x_1^2 + x_2^2 + \cdots + x_k^2$.)

(5) Any ordered field has an operation $S(x) = x + 1$, and for this operation $x < S(x)$.

(6) In any ordered field, $0 < S(0) < SS(0) < SSS(0) < \cdots$, and all elements of this sequence are distinct.

(1) Any field satisfies $x \cdot 0 = 0$ for any x.

We have $0 = 0 + 0$, since 0 is the additive identity element. Now multiply this equality by x and use the distributive law

$$\underline{x \cdot 0} = x \cdot (0 + 0) = \underline{x \cdot 0 + x \cdot 0}.$$

Now add $-(x \cdot 0)$ to the left and right sides. On the left we get $(-(x \cdot 0)) + (x \cdot 0) = 0$ by the Inverse Law for $+$. On the right we get

$$\begin{aligned} -(x \cdot 0) + (x \cdot 0 + x \cdot 0) &= (-(x \cdot 0) + x \cdot 0) + x \cdot 0, && \text{Assoc. Law, } + \\ &= 0 + x \cdot 0, && \text{Inverse Law, } - \\ &= x \cdot 0, && \text{Unit Law, } 0 \end{aligned}$$

Altogether this shows that $0 = x \cdot 0$.

(2) Any field satisfies $-(-x) = x$ for any x.

Since $-x$ is the additive inverse of x, we have $(-x) + x = 0$. Since $-(-x)$ is the additive inverse of $-x$, we have $(-x) + (-(-x)) = 0$. Hence $(-x) + x = 0 = (-x) + (-(-x))$. Adding x to the left and right sides of this equality we get on the left $x + ((-x) + x) = (x + (-x)) + x = 0 + x = x$, while on the right we get $x + ((-x) + (-(-x))) = (x + (-x)) + (-(-x)) = 0 + (-(-x)) = -(-x)$. Altogether this yields $x = -(-x)$.

(3) Any field satisfies $(-x)^2 = x^2$ for any x.

Multiplying $(x + (-x)) = 0$ by x yields $x^2 + x(-x) = 0$, or $x^2 = -(x(-x))$. Multiplying $(x + (-x)) = 0$ by $-x$ yields $x(-x) + (-x)^2 = 0$, or $(-x)^2 = -(x(-x))$. Hence $x^2 = (-x)^2$.

(4) In an ordered field,

(a) If $0 < x$, then $0 < x^2$.

If $0 < x$, then we can multiply both sides of $0 < x$ by the positive element x to obtain $x \cdot 0 < x \cdot x$, by the compatibility of order with multiplication by positive elements. But $x \cdot 0 = 0$ by Part (1). The expression x^2 is an abbreviation for $x \cdot x$, so we get $0 < x^2$.

(b) If $x < 0$, then $0 < -x$.

Add $-x$ to both sides of $x < 0$ to obtain $0 < -x$. (This uses the compatibility of order with addition and the Inverse Law for $-$.)

(c) If $x \neq 0$, then $0 < x^2$.

By the Law of Trichotomy, if $x \neq 0$, then $0 < x$ or $x < 0$. If $0 < x$ holds, then $0 < x^2$ by Part 4(a). If $x < 0$, then $0 < -x$ by Part 4(b). But from $0 < -x$ we derive $0 < (-x)^2$ by Part 4(a). Now $(-x)^2 = x^2$ by Part 3. Hence we get $0 < x^2$ in either case $0 < x$ or $x < 0$.

(d) Sums of nonzero squares are positive. (That is, $0 < x_1^2 + x_2^2 + \cdots + x_k^2$.)

In Part 4(c) we showed that nonzero squares are positive: $0 < x_i^2$. By the compatibility of order with addition, sums of positive elements are positive, so $0 < x_1^2 + \cdots + x_k^2$.

(5) Any ordered field has an operation $S(x) = x + 1$, and for this operation $x < S(x)$.

Ordered fields always have 1 and $=$, and they can be composed to create the operation $S(x) = x + 1$. Now, we must have $0 < 1$, since 1 is a nonzero square (Part 4(c)). Add x to both sides to obtain $x + 0 < x + 1$. This may be rewritten as $x < S(x)$ using Part 1 and the definition of S.

(6) In any ordered field, $0 < S(0) < SS(0) < SSS(0) < \cdots$, and all elements of this sequence are distinct.

We get $0 < S(0) < SS(0) < SSS(0) < \cdots$ by repeated use of Part 5. These elements are distinct by the irreflexivity of the strict order.