Practice with the Monotone Convergence Theorem!

(1) What is $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$?

First we have to decide what "$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$" means. Define a sequence

$$a_0 = \sqrt{1}, \quad a_1 = \sqrt{1+\sqrt{1}}, \quad a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}},$$

and continue by defining $a_{n+1} = \sqrt{1+a_n}$. I assert that the only reasonable meaning of "$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$" is that it is the limit of the sequence

$$(a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, \ldots) = \left(\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \ldots\right).$$

Here, the sequence $(a_n)_{n\in\mathbb{N}}$ is defined by recursion by stating that

$$\begin{aligned} a_0 &= 1 \qquad\qquad \text{and} \\ a_{n+1} &= \sqrt{1+a_n}. \end{aligned}$$

The numerical values of the terms in our sequence are approximately $(1, 1.414, 1.554, \ldots)$. It looks like it might be a monotone increasing sequence, so the MCT might prove convergence of this sequence. Let's postpone the MCT proof of convergence, and compute the limit under the assumption that the sequence converges.

Claim. If $(a_n)_{n\in\mathbb{N}}$ converges to limit L, then $L = \phi = \frac{1+\sqrt{5}}{2}$ = the Golden Ratio. (The numerical value is approximately 1.618).

Proof of Claim. Assume that $\lim a_n = L$. Square both sides of $a_{n+1} = \sqrt{1+a_n}$ to eliminate the square root symbol: we get $a_{n+1}^2 = 1 + a_n$. Apply the Algebraic Limit Theorem to both sides of this expression to obtain $L^2 = 1 + L$, which may be rewritten $L^2 - L - 1 = 0$. This shows that if $\lim a_n = L$, then L must be a root of the quadratic equation $x^2 - x - 1 = 0$. The roots of this quadratic are $x = \frac{1+\sqrt{5}}{2} = \phi$ and $x = \frac{1-\sqrt{5}}{2} = -1/\phi$. The first is positive and the second is negative. Since L is the limit of a sequence of positive terms, The Order Limit Theorem guarantees that $L \geq 0$, so $L \neq -1/\phi$, and it follows that $L = \phi$. $\square$

Despite the last sentence of the preceding proof, we have NOT proven that $L = \phi$ yet! We have only shown that IF $(a_n)_{n\in\mathbb{N}}$ converges, then its limit must be $L = \phi$. To complete the work, we must argue that $(a_n)_{n\in\mathbb{N}}$ converges. For this we will appeal to the MCT (Monotone Convergence Theorem). That is, we will prove that $(a_n)_{n\in\mathbb{N}}$ is monotone increasing and bounded above.

[Note that, if it is true that $(a_n)_{n\in\mathbb{N}}$ is monotone increasing and bounded above, then we have shown that $\lim_{n\to\infty} a_n = \phi$ must hold, hence ϕ is a good candidate for an upper bound of the sequence.]

Claim. $1 \leq a_n < a_{n+1} \leq \phi$ for all n.

Proof of Claim. This is by induction on n.
Base case, $n = 0$: Here $a_0 = 1, a_1 = \sqrt{2} \approx 1.414$, and $\phi \approx 1.618$, so

$$1 \leq a_0 < a_1 \leq \phi.$$

Inductive step. Assume that $1 \leq a_n < a_{n+1} \leq \phi$. Then

$$\begin{array}{ccccccccccl}
 & 1 & \leq & a_n & < & a_{n+1} & \leq & \phi, & & & \text{Assumption} \\
 & 1+1 & \leq & 1+a_n & < & 1+a_{n+1} & \leq & 1+\phi, & & & \text{Add 1} \\
 & \sqrt{1+1} & \leq & \sqrt{1+a_n} & < & \sqrt{1+a_{n+1}} & \leq & \sqrt{1+\phi}, & & & \text{Take square root} \\
 & \parallel & & \parallel & & \parallel & & \parallel & & & \\
1 \leq & \sqrt{2} & \leq & a_{n+1} & < & a_{n+2} & \leq & \phi. & & & \text{Desired conclusion}
\end{array}$$

Some facts used when moving from one line to the next:

(1) that the square root function on the positive real line preserves the strict order. This follows from the fact that it is the inverse of the squaring function, which preserves the strict order on the positive real line. (Proof: The axioms of ordered fields imply that if $0 < x < y$, then $xx < xy$ and $xy < yy$. By transitivity, $xx < yy$.)
(2) that $a_{n+1} = \sqrt{1+a_n}$ holds for every n. This is part of the recursive definition of $(a_n)_{n\in\mathbb{N}}$.
(3) that $\phi = \sqrt{1+\phi}$ holds. This is because ϕ is a root of $x^2 - x - 1 = 0$ (hence $\phi^2 = \phi + 1$, hence $\phi = \sqrt{1+\phi}$). $\square$

We also discussed the following problem.

(2) What is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}} \qquad ?$$

The only reasonable interpretation of this question is, What is the limit of the sequence

$$
\begin{aligned}
(b_n)_{n\in\mathbb{N}} &= (b_0, b_1, b_2, \ldots) \\
&= \left(1, 1+\tfrac{1}{1}, 1+\tfrac{1}{1+\frac{1}{1}}, \ldots\right) \\
&= (1,\ 2,\ 3/2,\ 5/3,\ 8/5, \ldots).
\end{aligned}
$$

The b_n-sequence is defined by the recursion

$$
\begin{aligned}
b_0 &= 1, \\
b_{n+1} &= 1 + 1/b_n.
\end{aligned}
$$

It is easy to see that, if $\lim_{n\to\infty} b_n = L$, then by the ALT we must have $\lim b_{n+1} = 1 + 1/(\lim b_n)$, or $L = 1 + 1/L$. This leads to $L^2 = L + 1$, and then $L^2 - L - 1 = 0$. Since L is a limit of positive terms, we must have that $L = \phi$, just as in Problem (1). Altogether this means that IF $\lim b_n$ exists, then the limit must be ϕ. Can we use the MCT to prove the limit exists?

In class we proved that $\lim_{n\to\infty} b_n$ DOES exist. The main steps in the argument were the following:

(1) The Fibonacci sequence is involved. This sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, and its first few values are
$$(F_0, F_1, F_2, \ldots) = (0, 1, 1, 2, 3, 5, 8, \ldots).$$
It is easy to see that $(F_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence of integers, and that $F_n \geq n$ when $n \geq 5$. By induction, it is not hard to show that this sequence grows exponentially fast ($F_{n+2} \geq (3/2)^n$ for $n \geq 0$). We explained why $b_n = F_{n+2}/F_{n+1}$.

(2) The subsequence of even terms of $(b_n)_{n\in\mathbb{N}}$ is monotone increasing and bounded above by ϕ, and the subsequence of odd terms of $(b_n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded below by ϕ.

(3) By Catalan's Identity, $|b_{n+1} - b_n| = \frac{1}{F_{n+2}F_{n+1}}$, which goes to zero. This is enough to prove that $(b_n)_{n\in\mathbb{N}}$ is bounded, and that the limit of the even subsequence is the same as that of the odd subsequence, and hence it is the limit of the whole sequence.