Different types of limits.

We have learned about limits of sequences. This can be used to define other kinds of limits.

(1) Decimal expansions.

When we write $\pi = 3.141592653589793\ldots$, we are saying that $\pi = \lim_{n\to\infty} a_n$ where

$$(a_0, a_1, a_2, a_3, \ldots) = (3, 3.1, 3.14, 3.141, \ldots) = \left(\frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \ldots\right)$$

is the sequence of best rational lower estimates of π whose denominators are powers of 10. Thus any decimal expansion is an abbreviation for a limit.

When we write $.9999\ldots = 1$, we mean that the sequence $(0, .9, .99, .999, \ldots)$ has limit 1, or that $\lim_{n\to\infty}(1 - 10^{-n}) = 1$.[1]

(2) Infinite series.

When we write $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, we are saying that $2 = \lim_{n\to\infty} s_n$ where s_n is the n-th partial sum of $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

$$\begin{aligned}
s_0 &= 1 & &= \tfrac{1}{1} & &= 2 - 1 \\
s_1 &= 1 + \tfrac{1}{2} & &= \tfrac{3}{2} & &= 2 - \tfrac{1}{2} \\
s_2 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} & &= \tfrac{7}{4} & &= 2 - \tfrac{1}{2^2} \\
s_n &= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n} & &= \tfrac{2^{n+1}-1}{2^n} & &= 2 - \tfrac{1}{2^n}
\end{aligned}$$

So $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 2.$

More generally, we study infinite series of the form $\sum a_i = a_0 + a_1 + a_2 + \cdots$.

(3) Infinite products.

The infinite product $\Pi_{n=0}^{\infty}(1 + a_n) = (1 + a_0)(1 + a_1)(1 + a_2)\cdots$ is defined to be the limit $\lim_{n\to\infty} p_n$ of the partial products

$$\begin{aligned}
p_0 &= (1 + a_0) \\
p_1 &= (1 + a_0)(1 + a_1) \\
p_2 &= (1 + a_0)(1 + a_1)(1 + a_2) \\
p_n &= (1 + a_0)(1 + a_1)(1 + a_2)\cdots(1 + a_n)
\end{aligned}$$

[1] You proved that $\lim_{n\to\infty}(10^{-n}) = 0$ on Quiz 5, so by the Algebraic Limit Theorem $\lim_{n\to\infty}(1 - 10^{-n}) = 1 - 0 = 1$. Thus, your quiz answer is the basis of a proper proof that $.9999\ldots = 1$.

(4) Continued fractions.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

means $\lim_{n\to\infty} b_n$, where

$$(b_0, b_1, b_2, \ldots) = \left(a_0, a_0 + \frac{1}{a_1}, a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}, \ldots \right).$$

(5) Nested roots.

$\sqrt{a_0 + \sqrt{a_1 + \sqrt{a_2 + \cdots}}}$ means $\lim_{n\to\infty} b_n$ where

$$(b_0, b_1, b_2, \ldots) = \left(\sqrt{a_0}, \sqrt{a_0 + \sqrt{a_1}}, \sqrt{a_0 + \sqrt{a_1 + \sqrt{a_2}}}, \ldots \right).$$