Practice with Cauchy Sequences!

This handout concerns

Theorem 1. *If $(a_i)_{i\in\mathbb{N}}$ converges, then it is a Cauchy sequence.*

(1) Rewrite the theorem in the form

Theorem 2. *A implies B.*

Where A and B are formal sentences.

$A =$

$B =$

(2) Give a winning strategy for the appropriate quantifier for the proof of the theorem in Problem 1. (Hint: you are proving the Conclusion B, but you are allowed to reference Hypothesis A in the proof.)

(3) Prove that your strategy in Problem 2 is a winning strategy.

(4) Rewrite your proof of the theorem in a way that does not explicitly refer to ∀belard or ∃loise.

Some Hints!

(1) $A = (\exists L)(\forall \varepsilon > 0)(\exists N)(\forall i)((i > N) \to (|a_i - L| < \varepsilon))$,
$B = (\forall \varepsilon > 0)(\exists N)(\forall i)(\forall j)((j > i > N) \to (|a_i - a_j| < \varepsilon))$

(2) The appropriate quantifier is $\exists$.

Strategy:

- $\forall$ chooses ε — May assume $\varepsilon > 0$
- $\exists$ chooses N — HOW?
- $\forall$ chooses i — May assume $i > N$
- $\forall$ chooses j — May assume $j > i$

To explain HOW $\exists$ chooses N, she plays Game A against her earlier self assuming the role of $\forall$, thereby forcing her earlier self to reveal her Game-A strategy.

So, assume that Present Day-$\forall$ chooses $\varepsilon > 0$ in Game B. Present Day-$\exists$ takes this value of ε back to the past and challenges Earlier-$\exists$ to Game A. Earlier-$\exists$ chooses L in Game A. Present Day-$\exists$, pretending to be Earlier-$\forall$, chooses the Game-A value of ε to be ONE HALF the Game-B value of ε (that is, $\varepsilon/2$). Earlier-$\exists$ chooses N which can force a win in Game A. Present Day-$\exists$ takes this value for N back to the present day and chooses it for the value of N in Game B.

(3) Why is this Game-B strategy a winning strategy?

The strategy guarantees that if $j > i > N$ in Game B, then $|a_i - L| < \varepsilon/2$ and $|a_j - L| < \varepsilon/2$ in Game A. Thus

$$|a_i - a_j| \leq |a_i - L| + |L - a_j| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

in Game B, as desired.