Notes on Cardinal Arithmetic.

Defns. cofinality, regular and singular cardinals.

Theorem 1. (1) *$cf(0) = 0$. The cofinality of a successor ordinal is 1. The cofinality of an infinite limit ordinal is an infinite cardinal.*
(2) *$cf(cf(\alpha)) = cf(\alpha)$. ($cf(\alpha)$ is regular cardinal.)*
(3) *$\aleph_{\alpha+1}$ is regular.*

Theorem 2. *If λ is infinite, $1 \leq \kappa_\alpha$ for all α, and $\kappa = \sup_{\alpha<\lambda}(\kappa_\alpha)$, then $\sum_{\alpha<\lambda} \kappa_\alpha = \kappa \cdot \lambda$.*

Theorem 3. *If λ is an infinite cardinal, $(\kappa_\alpha)_{\alpha<\lambda}$ is a nondecreasing λ-sequence of cardinals of size at least 2, and $\kappa = \sup_{\alpha<\lambda}(\kappa_\alpha)$, then $\prod_{\alpha<\lambda} \kappa_\alpha = \kappa^\lambda$.*

Theorem 4. *(König's Theorem) If $\kappa_i < \lambda_i$ for all $i \in I$, then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.*

Corollary 5. *If κ is infinite, then $\kappa^{cf(\kappa)} > \kappa$.*

Theorem 6. *Let κ and λ be infinite cardinals, and let $\tau = \sup_{\alpha<\kappa}(|\alpha|^\lambda)$.*

$$\kappa^\lambda = \begin{cases} 2^\lambda & \text{if } \kappa \leq 2^\lambda \\ \kappa \cdot \tau & \text{if } \lambda < cf(\kappa) \\ \tau & \text{if not in the above cases and } \alpha \mapsto |\alpha|^\lambda \text{ is eventually constant below } \kappa \\ \kappa^{cf(\kappa)} & \text{otherwise.} \end{cases}$$

Practice problems.

(1) Consider the ω_1-sequence of terms of the sequence that begins with the terms of $(\aleph_n)_{n\in\omega}$ and continues with ω_1 terms equal to 2.
  (a) Is it possible to rearrange the factors so that they form a nondecreasing ω_1-sequence?
  (b) Evaluate the product of all terms.
(2) If $\kappa \geq 2$ and $\lambda \geq \aleph_0$, show that $\mathrm{cf}(\kappa^\lambda) > \lambda$.
(3) Show that $|\mathbb{R}|$ has uncountable cofinality.
(4) Show that if GCH holds, then $\kappa^+ = \kappa^{\mathrm{cf}(\kappa)}$ for all infinite κ.