A divergent series with bounded partial sums!

Let θ be any angle. The series $\sum_{k=1}^{\infty} \cos(k\theta) = \cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cdots$ diverges by the Nth Term Test for Divergence. But if θ is not an even integer multiple of π, then this series has bounded partial sums.

To see this, we will use the Angle Addition Formula from trigonometry, which is:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha).$$

If we substitute $\alpha = k\theta$ and $\beta = (1/2)\theta$ in this formula we get

$$\sin(k\theta + (1/2)\theta) = \sin(k\theta)\cos((1/2)\theta) + \sin((1/2)\theta)\cos(k\theta).$$

Repeating this using $\alpha = k\theta$ and $\beta = (-1/2)\theta$ we get

$$\sin(k\theta - (1/2)\theta) = \sin(k\theta)\cos((1/2)\theta) - \sin((1/2)\theta)\cos(k\theta).$$

Subtracting the second result from the first leads to

$$\sin((k + 1/2)\theta) - \sin((k - 1/2)\theta) = 2\sin((1/2)\theta)\cos(k\theta).$$

If you sum both sides from $k = 1$ to $k = n$ you get

$$\sin((n + 1/2)\theta) - \sin((1/2)\theta) = 2\left(\sin((1/2)\theta)\left(\sum_{k=1}^{n} \cos(k\theta)\right)\right).$$

On the left hand side, we used telescopic cancellation.

Solving for $\sum_{k=1}^{n} \cos(k\theta)$ we get

$$s_n = \sum_{k=1}^{n} \cos(k\theta) = \frac{\sin((n + 1/2)\theta) - \sin((1/2)\theta)}{2\sin((1/2)\theta)} = \frac{\sin((n + 1/2)\theta)}{2\sin((1/2)\theta)} - \frac{1}{2}.$$

The denominator of the first fraction on the right hand side is not zero unless θ is an even integer multiple of π. Thus, the n-th partial sum $s_n = \sum_{k=1}^{n} \cos(k\theta)$ satisfies

$$|s_n| \leq \left|\frac{\sin((n + 1/2)\theta)}{2\sin((1/2)\theta)} - \frac{1}{2}\right| \leq \left|\frac{\sin((n + 1/2)\theta)}{2\sin((1/2)\theta)}\right| + \frac{1}{2} \leq \frac{1}{|2\sin(\theta/2)|} + \frac{1}{2}.$$

This upper estimate on $|s_n|$ depends on θ but not on n.

For example, if $\theta = .1$ radian, then

n	s_n	$\frac{1}{\lvert 2\sin(\theta/2)\rvert} + \frac{1}{2}$
1	.9950041653	10.50416788
10	8.177847573	10.50416788
20	8.377322108	10.50416788
30	.4150276369	10.50416788
40	-8.388539025	10.50416788
50	-9.939419291	10.50416788
60	-2.811740985	10.50416788
70	6.441341296	10.50416788
80	9.312586411	10.50416788
90	3.162184830	10.50416788
100	-6.355212600	10.50416788