The blancmange function. (Some hints in blue!)

The first example of a continuous, nowhere differentiable function was given by Karl Weierstrass (1872). A simpler example was given later by Teiji Takagi (1901), which is now called the blancmange[1] function (or curve) or else the Takagi function (or curve).

To define it, let $h(x) = \inf\{|x-n| \mid n \in \mathbb{Z}\}$ be the sawtooth function of period 1. The blancmange function is

$$B(x) = h(x) + \frac{1}{2}h(2x) + \frac{1}{4}h(4x) + \cdots = \sum_{k=0}^{\infty} \frac{1}{2^k} h(2^k x).$$

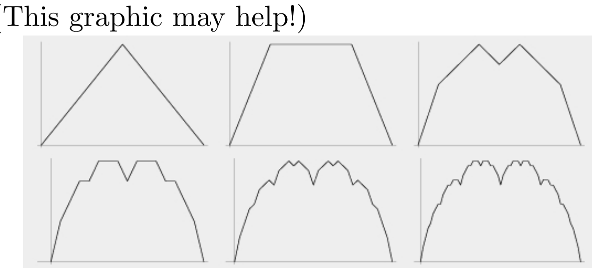
That is, $B(x) = \sum_{k=0}^{\infty} h_k(x)$ where $h_k(x) = \frac{1}{2^k} h(2^k x)$.

General theorems from Chapter 6 can be applied to prove that the blancmange function is continuous (Weierstrass M-Test and Uniform Limit Theorem). Here we argue that it is differentiable nowhere.

We will refer to *dyadic rationals*, which are rational numbers which have a representation as $\frac{m}{2^k}$ whose numerator is an integer and whose denominator is a positive integer power of 2. (For the purposes of this handout, say that $\frac{m}{2^k}$ is a *weight-k* representation of a dyadic rational.)

(1) Draw h_0, h_1, h_2 on the same coordinate system. Use different colors for different h_n's if you can. On a different coordinate system, draw the first few partial sums for $B(x)$:
$h_0(x), \quad h_0(x) + h_1(x), \quad h_0(x) + h_1(x) + h_2(x)$.

(This graphic may help!)



[1] "blancmange" refers to a white, puddinglike dessert made of sugar, cream, gelatin, and spices.

(2) Convince yourself why, if $n \geq k$, $h_n(\frac{m}{2^k}) = 0$.

Hint: Look back at your drawings. Convince yourself that the zeros of h_0 are the integers, which are the numbers of the form $\frac{m}{2^0}$. Then convince yourself that the zeros of h_1 are the half-integers, $\frac{m}{2^1}$. Then convince yourself that the zeros of h_2 are the quarter-integers, $\frac{m}{2^2}$. Can you think of an explanation why the zeros of h_k are the 2^kth-integers, $\frac{m}{2^k}$? Finally, convince yourself that the zeros of h_k are also zeros of $h_{k+1}, h_{k+2}, \ldots$.

(3) Convince yourself why, if $a_k := \frac{m}{2^k}$ and $b_k := \frac{m+1}{2^k}$ are consecutive dyadic rational of weight k, and $n < k$, the function $h_n(x)$ is linear on the the interval $[a_k, b_k]$, and that this linear function has slope $+1$ or -1. Moreover, this slope equals the derivative of $h_n(x)$ from the right at any point $c \in [a_k, b_k)$. (Write this slope as $h_n^+(c)$.)

Hint: Examine your drawings from the first part of this worksheet for $k = 2$. Look at two consecutive zeros for h_2. Now you are being asked what the earlier functions h_1 and h_0 are doing between these two points. It should be clear that h_1 and h_0 are linear of slope ± 1 between two consecutive zeros of h_2. Will this kind of statement be true for higher k?

(4) We wish to show that $B(x)$ is not differentiable at $x = c$ for arbitrarily chosen c. For this purpose, choose some $c \in \mathbb{R}$ which will remain fixed for the rest of this worksheet.

For this part, show that, for any weight k, it is possible to find consecutive dyadic rational a_k and b_k of weight k such that $a_k \leq c < b_k$.

Hint: If you can explain why any real number lies between two consecutive integers, then you have solved this part for $k = 0$. Now, given any c and k, find an integer m such that the real number $2^k c$ lies between m and $m + 1$. Let $a_k = \frac{m}{2^k}$ and $b_k = \frac{m+1}{2^k}$. This produces consecutive dyadic rationals a_k, b_k of weight k such that $a_k \leq c < b_k$.

(5) For a_k, c, b_k as in the last part, explain these equalities:

$$\frac{B(b_k) - B(a_k)}{b_k - a_k} = \frac{h_0(b_k) - h_0(a_k)}{b_k - a_k} + \cdots + \frac{h_{k-1}(b_k) - h_{k-1}(a_k)}{b_k - a_k} = h_0^+(c) + \cdots + h_{k-1}^+(c).$$

Hint: The first equality uses the definition $B(x) = \sum_{k=0}^{\infty} h_k(x)$ and the fact, from Part (2) above, that $h_n(\frac{m}{2^k}) = 0$ for $n \geq k$.

The second equality uses Part (3) above. The confusing part here is might be the notation $h_n^+(c)$. Recall that this is meant to be the derivative of h_n from the right at $x = c$. That is, it is $\lim_{x \to c^+} \frac{h_n(x) - h_n(c)}{x - c}$. You don't have to compute this, just remember that if $n < k$ then h_n is linear on $[a_k, b_k]$ with slope $h_n^+(c) \in \{+1, -1\}$ for any $c \in [a_k, b_k)$. This is enough to show that $\frac{h_n(a_k) - h_n(b_k)}{a_k - b_k} = h_n^+(c)$.

(6) Argue that, if $B'(c)$ existed, then the infinite series $\sum_{k=0}^{\infty} h_k^+(c)$ would have to converge to $B'(c)$.

Hint: This part looks like we just apply "lim" to the previous part, but it is not quite that easy. (Well, no, it is that easy, but some justification is required.)

We know that the sequence (a_k) approaches c from the left and (b_k) approaches c from the right. So, if $B'(c)$ exists, we would like to claim that $\lim_{k \to \infty} \frac{B(b_k) - B(a_k)}{b_k - a_k} = B'(c)$. But that is not how $B'(c)$ is defined, so we need justification here.

Assume that $B'(c)$ exists. Then $\lim_{x \to c} \frac{B(x) - B(c)}{x - c} = B'(c)$. This forces

$$\lim_{k \to \infty} \frac{B(a_k) - B(c)}{a_k - c} = B'(c) = \lim_{k \to \infty} \frac{B(c) - B(a_k)}{c - a_k}$$

since the sequence (a_k) approaches c. Similarly $\lim_{k \to \infty} \frac{B(b_k) - B(c)}{b_k - c} = B'(c)$.

Now,

$$\frac{B(b_k) - B(a_k)}{b_k - a_k} = \left(\frac{b_k - c}{b_k - a_k}\right)\left(\frac{B(b_k) - B(c)}{b_k - c}\right) + \left(\frac{c - a_k}{b_k - a_k}\right)\left(\frac{B(c) - B(a_k)}{c - a_k}\right),$$

and the fractions $\frac{b_k - c}{b_k - a_k}$ and $\frac{c - a_k}{b_k - a_k}$ are nonnegative real numbers that sum to 1. This shows that $\frac{B(b_k) - B(a_k)}{b_k - a_k}$ is a weighted average of the values $\frac{B(b_k) - B(c)}{b_k - c}$ and $\frac{B(c) - B(a_k)}{c - a_k}$, hence $\frac{B(b_k) - B(a_k)}{b_k - a_k}$ must lie between $\frac{B(b_k) - B(c)}{b_k - c}$ and $\frac{B(c) - B(a_k)}{c - a_k}$. Since the two outer values both approach $B'(c)$ as $k \to \infty$, the inner value $\frac{B(b_k) - B(a_k)}{b_k - a_k}$ must also approach $B'(c)$.

(7) Explain why $B'(c)$ cannot exist.

Hint: The logic is this: if $B'(c)$ exists, then $\lim_{k \to \infty} \frac{B(b_k) - B(a_k)}{b_k - a_k}$ must exist, hence $\sum_{k=0}^{\infty} h_k^+(c)$ must exist. But $\sum_{k=0}^{\infty} h_k^+(c)$ does not exist. (Why?)