Practice!

(1) State the theorem.

  (a) Alternating Series Test for Convergence/Dirichlet's Test for Convergence

  (b) Riemann Rearrangement Theorem

  (c) Heine-Borel Theorem

  (d) Heine-Cantor Theorem

  (e) Extreme Value Theorem

  (f) Intermediate Value Theorem

  (g) Interior Extremum Theorem

  (h) Darboux's Theorem

  (i) Rolle's Theorem/Mean Value Theorem

  (j) Uniform Limit Theorem

  (k) Weierstrass M-Test

  (l) Clairaut's Theorem, Leibniz Integral Rule, Fubini's Theorem

  (m) Dini's Theorem

  (n) Lebesgue's Characterization of Riemann Integrable Functions

(2) Define the terms.
  (a) absolutely convergent sequence, conditionally convergent sequence, radius of convergence
  (b) topology/topological space, open and closed sets, interior and closure, neighborhood, limit point, boundary point, interior point, isolated point, discrete topology, trivial topology, cofinite topology, metric/metric topology, ε-ball.
  (c) compact set
  (d) connected subset
  (e) continuous function, uniformly continuous function
  (f) derivative
  (g) pointwise convergence, uniform convergence, absolutely convergent sequence of functions
  (h) Riemann integral, partition
  (i) measure zero set

(3) Give an example if possible, or an explanation if not possible, of
  (a) A conditionally convergent series.
  (b) An unbounded compact set.
  (c) An unbounded connected set.
  (d) A connected subset of $\mathbb{R}$ whose complement is also connected.
  (e) A function defined everywhere, but continuous at only one point.
  (f) A function continuous everywhere, but differentiable at only one point.
  (g) A pair of functions f, g and a subset A such that $f' = g'$ on A and $f(a) = g(a)$ for some $a \in A$, but $f \neq g$ on A.
  (h) A series $\sum f_n$ that converges pointwise, but not uniformly on $[-1, 1]$.
  (i) A series $\sum f_n$ of continuous functions that converges everywhere on $[-1, 1]$ except at $x = 0$.
  (j) A partition of 2 cells of the interval $[0, 1]$ for which $U(f, P) - L(f, P) < 1/100$, where f is the function defined by $f(0) = f(1) = 1$, and $f(x) = 0$ if $x \neq 0, 1$.
  (k) A partition of 3 cells of the interval $[0, 1]$ for which $U(f, P) - L(f, P) < 1/100$, where f is the function defined by $f(0) = f(1) = 1$, and $f(x) = 0$ if $x \neq 0, 1$.
  (l) A function f, integrable on $[a, b]$, but $(\int_a^x f)' \neq f$.
  (m) A function f, differentiable on $[a, b]$, but f' is not integrable on $[a, b]$.
  (n) A sequence of functions (f_n), bounded on $[0, 1]$, where no member is integrable on $[0, 1]$, but the sequence converges uniformly to an integrable function on $[0, 1]$.
  (o) An uncountable set of measure zero.

(4) True or False? If False, explain why.

(a) Absolute convergence is stronger than conditional convergence, so any absolutely convergent sequence is also conditionally convergent.

(b) An open subset of $\mathbb{R}$ that contains every rational number must be all of $\mathbb{R}$.

(c) The Nested Interval Property remains true if "closed interval" is replace by "closed set".

(d) The Nested Interval Property remains true if "closed interval" is replace by "compact set".

(e) If f is continuous and O is open, then $f(O)$ is open.

(f) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set O such that $f(O) = \mathbb{Q}$.

(g) If f is differentiable on an interval I, and $f'(x) \neq 0$ on I, then f is monotone on I.

(h) The Weierstrass M-test can be used to prove that a series converges uniformly.

(i) Dini's Theorem can be used to prove that a sequence of functions converges uniformly.

(j) $\lim_{x\to a} \lim_{y\to b} f(x, y) = \lim_{y\to b} \lim_{x\to a} f(x, y)$.

(k) The set of discontinuities of the absolute value function has measure zero.

(l) The set of discontinuities of the Dirichlet function has measure zero.

(m) A union of countably many measure zero sets has measure zero.

(n) A union of uncountably many measure zero sets has measure zero.

(5) Explain why $\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)' = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$ for any x.

(6) Suppose $K \subseteq \mathbb{R}$ is such that $f(K)$ is compact for every continuous function $f : \mathbb{R} \to \mathbb{R}$. Must K be compact?

(7) Show that any constant sequence $(f)_{n\in\mathbb{N}} = (f, f, f, \ldots)$ converges uniformly on any interval.

(8) Show that the sequence

$$\left(1,\ 1+x,\ 1+x+\frac{x^2}{2!},\ 1+x+\frac{x^2}{2!}+\frac{x^3}{3!},\ \cdots\right)$$

converges uniformly on $[-c, c]$ for any $c \in \mathbb{R}$.

(9) Show that if $(f_n) \to f$ uniformly, and each f_n is bounded on $[a, b]$, then f is bounded on $[a, b]$.

(10) What property of $(f_n)_{n\in\mathbb{N}}$ and f is being expressed by the following sentence?

$$(\forall c)(\forall \varepsilon > 0)(\exists N)(\forall x)((n > N) \to |f_n(c) - f(c)| < \varepsilon)$$

(11) Dini's Theorem asserts that if (f_n) converges monotonically to f on $[a, b]$, and all f_n and f are continuous on $[a, b]$, then the convergence $(f_n) \to f$ is uniform. Write each of the following phrases formally. (Double check that your quantifier structure is correct!)

(a) "the sequence (f_n) increases monotonically on $[a, b]$"

(b) "(f_n) converges to f"

(c) "f_n and f are continuous on $[a, b]$"

(d) "(f_n) converges to f uniformly"

(12) Show that if f is integrable on $[a, b]$ and on $[b, c]$, then it is integrable on $[a, c]$.

(13) Show that if f is integrable on $[a, c]$ and $a < b < c$, then f is integrable on $[a, b]$ and $[b, c]$.

(14) Show that if (f_n) is a sequence of functions, bounded on $[0, 1]$, where each f_n has finitely many discontinuities, and (f_n) converges to f on $[0, 1]$, then
(a) if the convergence is uniform, then the set of discontinuities of f must have measure zero, but
(b) if the convergence is not uniform, then the set of discontinuities of f may not have measure zero.