Practice!

The goal is to prove the following

Theorem 1. *The continuous image of a compact set is compact.*

(In more detail, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact, then $f(K) = \{f(r) \mid r \in K\}$ is compact.)

Recall that we have three definitions for compactness for subsets $K \subseteq \mathbb{R}$:

(a) (In terms of limits.) Every sequence in K has a convergent subsequence, and the limit of this subsequence belongs to K.
(b) (In terms of the metric topology.) K is closed and bounded.
(c) (In purely topological terms.) Every open cover of K has a finite subcover.

We also have three definitions for the continuity of a function $f : \mathbb{R} \to \mathbb{R}$:

(a) (In terms of limits.) f preserves limits. ($f(\lim a_i) = \lim f(a_i)$.)
(b) (In terms of the metric topology.) f reflects open balls. (For all $r \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(r)) \subseteq B_\varepsilon(f(r))$.)
(c) (In purely topological terms.) f reflects open sets. (If $O \in \tau$ is open, then $f^{-1}(O)$ is open.)

Exercise. Provide type (a) and type (c) proofs of the theorem.

"Type (a)" Proof.

"Type (c)" Proof.