THEOREM 2.19. *Let* $\mathbf{A}$ *be an algebra with tolerances* S, S', T, T' *and congruences* α, α_i, β, δ, δ', δ_j. *The following are true.*

(1) (Monotonicity in the first two variables) *If* $\mathbf{C}(S,T;\delta)$ *holds and* $S' \subseteq S$, $T' \subseteq T$, *then* $\mathbf{C}(S',T';\delta)$ *holds.*
(2) $\mathbf{C}(S,T;\delta)$ *holds if and only if* $\mathbf{C}(\mathrm{Cg}^{\mathbf{A}}(S),T;\delta)$ *holds.*
(3) $\mathbf{C}(S,T;\delta)$ *holds if and only if* $\mathbf{C}(S,\delta\circ T\circ\delta;\delta)$ *holds.*
(4) *If* $T\cap\delta = T\cap\delta'$, *then* $\mathbf{C}(S,T;\delta) \iff \mathbf{C}(S,T;\delta')$.
(5) (Semidistributivity in the first variable) *If* $\mathbf{C}(\alpha_i,T;\delta)$ *holds for all* $i\in I$, *then* $\mathbf{C}(\bigvee_{i\in I}\alpha_i,T;\delta)$ *holds.*
(6) *If* $\mathbf{C}(S,T;\delta_j)$ *holds for all* $j\in J$, *then* $\mathbf{C}(S,T;\bigwedge_{j\in J}\delta_j)$ *holds.*
(7) *If* $T\cap\big(S\circ(T\cap\delta)\circ S\big)\subseteq\delta$, *then* $\mathbf{C}(S,T;\delta)$ *holds.*
(8) *If* $\beta\wedge\big(\alpha\vee(\beta\wedge\delta)\big)\leq\delta$, *then* $\mathbf{C}(\alpha,\beta;\delta)$ *holds.*
(9) *Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$. *If* $\mathbf{C}(S,T;\delta)$ *holds in* $\mathbf{A}$, *then* $\mathbf{C}(S|_{\mathbf{B}},T|_{\mathbf{B}};\delta|_{\mathbf{B}})$ *holds in* $\mathbf{B}$.
(10) *If* $\delta'\leq\delta$, *then the relation* $\mathbf{C}(S,T;\delta)$ *holds in* $\mathbf{A}$ *if and only if* $\mathbf{C}(S/\delta',T/\delta';\delta/\delta')$ *holds in* $\mathbf{A}/\delta'$.

PROOF. Item (1) follows from the fact that $M(S',T')\subseteq M(S,T)$.

For (2), $\mathbf{C}\big(\mathrm{Cg}^{\mathbf{A}}(S),T;\delta\big) \implies \mathbf{C}(S,T;\delta)$ follows from (1), since $S\subseteq\mathrm{Cg}^{\mathbf{A}}(S)$. For the reverse implication (and also for the proof of item (5)), we will argue that if S_i is a tolerance, $\mathbf{C}(S_i,T;\delta)$ holds for all $i\in I$, and $\alpha := \text{tr.cl.}\big(\bigcup_{i\in I}S_i\big)$, then $\mathbf{C}(\alpha,T;\delta)$. (To complete the proof of (2) we need this only when $|I|=1$, while in (5) we need it only when the S_i are congruences.)

Choose any matrix in $M(\alpha,T)$. If it is

$$\begin{bmatrix} p & q\\ r & s\end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}) & f(\mathbf{a},\mathbf{v})\\ f(\mathbf{b},\mathbf{u}) & f(\mathbf{b},\mathbf{v})\end{bmatrix},$$

then $\mathbf{a}$ is related to $\mathbf{b}$ by $\text{tr.cl.}\big(\bigcup_{i\in I}S_i\big)$. It is easy to see that there exist tuples $\mathbf{a}=\mathbf{a}_0\ S_{i_1}\ \mathbf{a}_1\ S_{i_2}\ \cdots\ S_{i_n}\ \mathbf{a}_n=\mathbf{b}$. These tuples determine matrices

$$\begin{bmatrix} p_k & q_k\\ p_{k+1} & q_{k+1}\end{bmatrix} := \begin{bmatrix} f(\mathbf{a}_k,\mathbf{u}) & f(\mathbf{a}_k,\mathbf{v})\\ f(\mathbf{a}_{k+1},\mathbf{u}) & f(\mathbf{a}_{k+1},\mathbf{v})\end{bmatrix} \in M(S_{i_{k+1}},T)\,.$$

We must show that $p\equiv_\delta q$ implies $r\equiv_\delta s$, so assume that $p\equiv_\delta q$. This is the same as $p_0\equiv_\delta q_0$, and so by induction (using that $\mathbf{C}(S_{i_k},T;\delta)$ holds for all k) we get that $p_k\equiv_\delta q_k$ for all k. Therefore $r=p_n\equiv_\delta q_n=s$. This completes the proofs of (2) and (5).

For (3), the implication $\mathbf{C}(S,\delta\circ T\circ\delta;\delta) \implies \mathbf{C}(S,T;\delta)$ follows from (1), since $T\subseteq\delta\circ T\circ\delta$. For the reverse implication, assume that

$\mathbf{C}(S,T;\delta)$ holds, that

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}) & f(\mathbf{a},\mathbf{v}) \\ f(\mathbf{b},\mathbf{u}) & f(\mathbf{b},\mathbf{v}) \end{bmatrix} \in M(S, \delta \circ T \circ \delta)\,,$$

and that $p \equiv_\delta q$. There exist tuples $\mathbf{u}'$ and $\mathbf{v}'$ such that $\mathbf{u}\ \delta\ \mathbf{u}'\ T\ \mathbf{v}'\ \delta\ \mathbf{v}$. The matrix

$$\begin{bmatrix} p' & q' \\ r' & s' \end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}') & f(\mathbf{a},\mathbf{v}') \\ f(\mathbf{b},\mathbf{u}') & f(\mathbf{b},\mathbf{v}') \end{bmatrix}$$

is an S,T-matrix. Moreover,

$$p' = f(\mathbf{a},\mathbf{u}')\ \delta\ f(\mathbf{a},\mathbf{u}) = p\ \delta\ q = f(\mathbf{a},\mathbf{v})\ \delta\ f(\mathbf{a},\mathbf{v}') = q'\,.$$

Since $\mathbf{C}(S,T;\delta)$ holds, it follows that $r' \equiv_\delta s'$. Hence

$$r = f(\mathbf{b},\mathbf{u})\ \delta\ f(\mathbf{b},\mathbf{u}') = r'\ \delta\ s' = f(\mathbf{b},\mathbf{v}')\ \delta\ f(\mathbf{b},\mathbf{v}) = s\,,$$

or $r \equiv_\delta s$. This establishes $\mathbf{C}(S, \delta \circ T \circ \delta; \delta)$.

For (4), recall that elements in the same row of an S,T-matrix are T-related. So if $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T)$, then since $T \cap \delta = T \cap \delta'$ we get that

$$p \equiv_\delta q \iff p \equiv_{T\cap\delta} q \iff p \equiv_{T\cap\delta'} q \iff p \equiv_{\delta'} q\,,$$

and

$$r \equiv_\delta s \iff r \equiv_{T\cap\delta} s \iff r \equiv_{T\cap\delta'} s \iff r \equiv_{\delta'} s\,.$$

Therefore the implication $p \equiv_\delta q \implies r \equiv_\delta s$ is equivalent to the implication $p \equiv_{\delta'} q \implies r \equiv_{\delta'} s$.

For (6), assume that $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T)$. If $p \equiv q \pmod{\bigwedge \delta_j}$, then $p \equiv q \pmod{\delta_j}$ for all j. Since $\mathbf{C}(S,T;\delta_j)$ holds for all j we get that $r \equiv s \pmod{\delta_j}$ for all j, or equivalently that $r \equiv s \pmod{\bigwedge \delta_j}$. This shows that $\mathbf{C}(S,T;\bigwedge_{j\in J}\delta_j)$ holds.

For (7), choose an S,T-matrix $M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Assume that $p \equiv_\delta q$. Since the elements in the same row of M are T-related and the elements in the same column are S-related, we have $r\ S\ p\ T\cap\delta\ q\ S\ s$. Moreover, $r\ T\ s$ since these elements belong to the same row. Together this yields that $r\ T \cap \big(S \circ (T\cap\delta) \circ S\big)\ s$. By the assumption in (7), this implies that $r \equiv_\delta s$. This proves (7).

For item (8), if $\beta \wedge \big(\alpha \vee (\beta\wedge\delta)\big) \leq \delta$, then $\beta \cap \big(\alpha \circ (\beta\cap\delta) \circ \alpha\big) \leq \delta$, so $\mathbf{C}(\alpha,\beta;\delta)$ holds by (7).

Item (9) holds because any instance of the implication in Definition 2.18 defining $\mathbf{C}(S|_{\mathbf{B}}, T|_{\mathbf{B}}; \delta|_{\mathbf{B}})$ in $\mathbf{B}$ is an instance of the implication defining $\mathbf{C}(S,T;\delta)$ in $\mathbf{A}$.

For item (10), it suffices to observe that, when $\delta' \leq \delta$,

$$\begin{bmatrix} p'/\delta' & q'/\delta' \\ r'/\delta' & s'/\delta' \end{bmatrix} \in M(S/\delta', T/\delta')$$

if and only if there exist $p \equiv_{\delta'} p'$, $q \equiv_{\delta'} q'$, $r \equiv_{\delta'} r'$, and $s \equiv_{\delta'} s'$ with

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S, T),$$

and that $p \equiv_\delta q \Leftrightarrow p'/\delta' \equiv_{\delta/\delta'} q'/\delta'$ and $r \equiv_\delta s \Leftrightarrow r'/\delta' \equiv_{\delta/\delta'} s'/\delta'$. $\square$

Definition 2.20. The **commutator** of S and T, denoted by $[S,T]$, is the least congruence δ such that $\mathbf{C}(S,T;\delta)$ holds. T is **abelian** if $[T,T] = 0$. An algebra $\mathbf{A}$ is **abelian** if its largest congruence is.

By Theorem 2.19 (6), the class of all δ such that $\mathbf{C}(S,T;\delta)$ holds is closed under complete meet, so there is a least such δ. This implies that $[S,T]$ exists for any two tolerances S and T.

It is a well known fact, easily derivable from the definitions, that $\mathbf{A}$ is abelian if and only if the diagonal of $A \times A$ is a class of a congruence of $\mathbf{A} \times \mathbf{A}$.

Definition 2.21. The **centralizer of T modulo δ**, denoted by $(\delta : T)$, is the largest congruence α on $\mathbf{A}$ such that $\mathbf{C}(\alpha, T; \delta)$ holds.

By Theorem 2.19 (5), the class of all α such that $\mathbf{C}(\alpha, T; \delta)$ holds is closed under complete join, so there is a largest such α. This implies that $(\delta : T)$ exists for every δ and T. By Theorem 2.19 (2), the centralizer $(\delta : T)$ contains every tolerance S such that $\mathbf{C}(S,T;\delta)$ holds.

2.6. Congruence Identities

If $\mathcal{V}$ is a variety of algebras, then any lattice identity that holds in the class $\{\mathbf{Con}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$ of congruence lattices of algebras in $\mathcal{V}$ is called a **congruence identity** of $\mathcal{V}$. The **congruence variety** of $\mathcal{V}$, denoted $\mathrm{CON}(\mathcal{V})$, is the subvariety of $\mathcal{L}$ generated by $\{\mathbf{Con}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$, or alternatively is the variety of lattices axiomatized by the congruence identities that hold in $\mathcal{V}$. Similarly, a lattice quasi-identity that holds in congruence lattices of members of $\mathcal{V}$ is a **congruence quasi-identity** of $\mathcal{V}$.

The following theorem will be used in several places in this monograph.

Theorem 2.22 (Cf. [**6**]). *Let Q be a quasi-identity satisfying (W). The class of varieties satisfying Q as a congruence quasi-identity is definable by a set of idempotent Maltsev conditions.*