THE GALOIS CONNECTION BETWEEN OPERATIONS AND RELATIONS

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The aim of these notes is to describe the Galois closed sets of operations and relations, allowing both operations and relations be of any finite or infinite arity.

1. Preliminaries

Cardinals will be denoted by Greek letters $\kappa, \lambda, \mu, \nu, \zeta$ and ordinals by Latin letters k, l, m, n. For any cardinal μ, the successor cardinal is denoted by μ^+.

Let A be a fixed set. A **ν-ary operation** on A is a function $A^\nu \to A$, while a **ν-ary relation** on A is a subset of A^ν. The set of all ν-ary operations on A will be denoted by $\mathrm{Op}_A^{(\nu)}$, and the set of all ν-ary relations on A will be denoted by $\mathrm{Rel}_A^{(\nu)}$. We will use the symbols $\mathrm{Op}_A^{(<\kappa)}$ and $\mathrm{Op}_A^{(<\infty)}$ to denote the graded set of all operations of arity $< \kappa$ on A, and the graded set (class) of all operations on A. The analogous sets of relations are denoted by $\mathrm{Rel}_A^{(<\kappa)}$ and $\mathrm{Rel}_A^{(<\infty)}$, respectively.

For an operation $f \in \mathrm{Op}_A^{(\nu)}$ and a relation $\rho \in \mathrm{Rel}_A^{(\mu)}$ we say that f **preserves** ρ, or ρ **is compatible with** f, and write $f \perp \rho$, if $f(\rho, \rho, \ldots) \subseteq \rho$. Equivalently, f preserves ρ iff ρ is a subalgebra of the algebra $\langle A; f\rangle^\mu$ iff f if a homomorphism $\langle A; \rho\rangle^\nu \to \langle A; \rho\rangle$ of relational structures. For any pair κ, λ where each one of κ, λ is either a cardinal or ∞, compatibility of operations and relations defines a Galois connection between subsets of $\mathrm{Op}_A^{(<\kappa)}$ and $\mathrm{Rel}_A^{(<\lambda)}$ as follows:

$$\mathrm{Op}_A^{(<\kappa)} \rightleftharpoons \mathrm{Rel}_A^{(<\lambda)}, \tag{1.1}$$

$$(\mathrm{Op}_A^{(<\kappa)} \supseteq)\ F \mapsto F^\perp = \{\rho \in \mathrm{Rel}_A^{(<\lambda)} : f \perp \rho \text{ for all } f \in F\},$$

$$\{f \in \mathrm{Op}_A^{(<\kappa)} : f \perp \rho \text{ for all } \rho \in R\} = R^\perp \mapsfrom R\ (\subseteq \mathrm{Rel}_A^{(<\lambda)}).$$

A set $C \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations is **Galois closed** if $C = C^{\perp\perp}$, or equivalently, if $C = R^\perp$ for some set $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$ of relations. Similarly, a set $K \subseteq \mathrm{Rel}_A^{(<\lambda)}$ of relations is **Galois closed** if $K = K^{\perp\perp}$, or equivalently, if $K = F^\perp$ for some set $F \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations. Our aim is to find operations on $\mathrm{Op}_A^{(<\kappa)}$ and $\mathrm{Rel}_A^{(<\lambda)}$ such that the Galois closed sets are exactly those subsets of $\mathrm{Op}_A^{(<\kappa)}$ and $\mathrm{Rel}_A^{(<\lambda)}$ that are closed under these operations.

Operations on $\mathrm{Op}_A^{(<\kappa)}$

- Distinguished elements (0-ary operations): They are the **projection operations** $p_n^{(\nu)} \in \mathrm{Op}_A^{(\nu)}$ $(n < \nu < \kappa)$ defined by
$$p_n^{(\nu)}(\overline{a}) = a_n \qquad \text{for all } \overline{a} = (a_j)_{j<\nu} \in A^\nu.$$
- **Composition**: For $f \in \mathrm{Op}_A^{(\mu)}$ $(\mu < \kappa)$ and $g_m \in \mathrm{Op}_A^{(\nu)}$ $(m < \mu < \kappa)$, their composition $f(g_0, g_1, g_2, \ldots) \in \mathrm{Op}_A^{(\nu)}$ is defined by
$$f(g_0, g_1, g_2, \ldots)(\overline{a}) = f(g_0(\overline{a}), g_1(\overline{a}), g_2(\overline{a}), \ldots) \qquad \text{for all } \overline{a} \in A^\nu.$$
- Local closure: Let $F \subseteq \mathrm{Op}_A^{(<\kappa)}$ and let λ be a cardinal or ∞. We will say that a ν-ary operation g is $(< \lambda)$**-locally in** F if for any set $B \subseteq A^\nu$ with $|B| < \lambda$ there exists a ν-ary operation $f \in F$ such that $g|_B = f|_B$. If F contains every operation that is $(< \lambda)$-locally in F, then F will be called $(< \lambda)$**-locally closed**. The phrase '**locally closed**' will be used to mean '$(< \omega)$-locally closed'.

A subset C of $\mathrm{Op}_A^{(<\kappa)}$ that contains all projection operations $p_n^{(\nu)}$ $(n < \nu < \kappa)$ and is closed under composition is called a **clone of** $(< \kappa)$**-ary operations on** A. If $\kappa = \omega$, then C is called a **clone on** A.

When working with relations, it will be convenient to allow the coordinates of a relation to be indexed by a set other than its arity (a cardinal). We now introduce some terminology and notation for this more general approach. If X is a set of cardinality μ, then every bijection $\varphi\colon \mu \to X$ induces a bijection $A^X \to A^\mu$, $r \mapsto r \circ \varphi$ and hence every set $\rho \subseteq A^X$ of functions $X \to A$ can be identified with the corresponding relation $\{r \circ \varphi : r \in \rho\} \subseteq A^\mu$. Different bijections $\varphi, \psi\colon \mu \to X$ yield relations in A^μ that differ by a fixed permutation $\psi^{-1} \circ \varphi$ of coordinates.

More generally, if X is a set of cardinality $|X| \le \mu$, an onto map $\varphi\colon \mu \to X$, $m \mapsto x_m$ will be called a μ**-multiset**, which can be thought of as an indexing of the elements of X by ordinals $< \mu$, or a collection $X_\varphi = \{x_m : m < \mu\}$ of μ elements, some of which are equal. For any function $r \in A^X$, we define the map $r \circ \varphi : \mu \to A$ $\big($or equivalently, the map $X_\varphi \to A$, $x_m \mapsto r(x_m)$ $(m < \mu)\big)$ to be the corresponding function from the μ-multiset X_φ to A, and let A^{X_φ} denote the set of all functions from the μ-multiset X_φ to A. Using the one-to-one function $A^X \to A^{X_\varphi}$, $r \mapsto r \circ \varphi$ we get for every relation $\rho \subseteq A^X$ a corresponding relation $\{r \circ \varphi : r \in \rho\} \subseteq A^{X_\varphi}$ which differs from ρ by repeating coordinates (as determined by the kernel of φ). If there is no danger of confusion, we will write X for the μ-multiset X_φ.

Now let Y be a set, and let $X = \{x_m : m < \mu\}$ be a μ-multiset in Y (i.e., $x_m \in Y$ for all $m < \mu$). For a function $f\colon Y \to A$ let $f|_X$ denote the restriction of f to the multiset X; that is, $f|_X : X = \{x_m : m < \mu\} \to A$, $x_m \mapsto f(x_m)$. For arbitrary relations $\rho \subseteq A^X$ and $\sigma \subseteq A^Y$ let

$$\pi_X(\sigma) = \{s|_X : s \in \sigma\} \qquad \text{and} \qquad \pi_Y^{-1}(\rho) = \{f \in A^Y : f|_X \in \rho\};$$

that is, $\pi_X(\sigma)$ is the **projection** of σ onto the μ-multiset X of coordinates, and $\pi_Y^{-1}(\rho)$ is the largest relation $\overline{\rho} \subseteq A^Y$ such that $\pi_X(\overline{\rho}) = \rho$.

Operations on $\mathrm{Rel}_A^{(<\lambda)}$

- **Definitions by primitive positive (p.p.) formulas**: Let $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$. A p.p. formula in R with free variables x_m, $m < \mu$ ($\mu < \lambda$), is a formula of the form

$$\exists (y)_{y \in Y \setminus X} \left(\bigwedge_{i \in I} \rho_i\big((y_{i\ell})_{\ell < \mu_i}\big) \right) \tag{1.2}$$

  where $Y = X \cup \bigcup_{i \in I} X_i$ is a set of variables, each ρ_i ($i \in I$) is a relation of arity $\mu_i\,(< \lambda)$ in $R \cup \{=_A\}$, $\mu_i \to X_i\,(\subseteq Y)$, $m \mapsto y_{im}$ are onto functions, and $\mu \to X\,(\subseteq Y)$, $m \mapsto x_m$ is a bijection. The μ-ary relation defined by (1.2) is

$$\rho = \{f|_X : f \in A^Y \text{ and } f|_{X_i} \in \rho_i \text{ for all } i \in I\} = \pi_X\left(\bigcap_{i \in I} \pi_Y^{-1}(\rho_i)\right),$$

  which is also called a **superposition** of the relations ρ_i ($i \in I$). Note that the cardinality of Y is allowed to be arbitrarily large (in particular, $|Y|$ is not bounded by λ). By the **length** of a p.p. formula we mean the number of symbols contained in the formula. So, the length of (1.2) is either finite or equal to $\sum_{i \in I} \mu_i$.

- Local closure: Let $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$ and let κ be a cardinal or ∞. We will say that a μ-ary relation ρ is $(< \kappa)$**-locally in** R if for any set $B \subseteq \rho$ with $|B| < \kappa$ there exists a μ-ary relation $\sigma \in R$ such that $B \subseteq \sigma \subseteq \rho$. If R contains every relation that is $(< \kappa)$-locally in R, then R will be called $(< \kappa)$**-locally closed**. The phrase '**locally closed**' will be used to mean '$(< \omega)$-locally closed'.

A subset K of $\mathrm{Rel}_A^{(<\lambda)}$ that is closed under definitions by p.p. formulas of *any* length is called a **relational clone of** $(< \lambda)$**-ary relations on** A. If $\kappa = \omega$, K is called a **relational clone on** A.

2. Galois Closed Sets of Operations and Relations

Theorem 2.1. *Let A be a set with at least two elements, and let each one of κ, λ be either an infinite cardinal or ∞. A set $C \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations is Galois closed in the Galois connection $\mathrm{Op}_A^{(<\kappa)} \rightleftharpoons \mathrm{Rel}_A^{(<\lambda)}$ if and only if C is a $(<\lambda)$-locally closed clone of $(<\kappa)$-ary operations on A.*

Proof. Let $C \subseteq \mathrm{Op}_A^{(<\kappa)}$. If C is Galois closed, then $C = R^{\perp}$ for some set $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$ of relations. To see that C is a $(<\lambda)$-locally closed clone of $(<\kappa)$-ary operations on A, it suffices to prove the following claim.

Claim 2.2. (a) *The projection operations preserve every relation ρ.*
(b) *If a μ-ary operation f and ν-ary operations g_m $(m < \mu)$ all preserve a relation ρ, then their composition $f(g_0, g_1, g_2, \dots)$ also preserves ρ.*
(c) *If all members of a set F of operations preserve a relation ρ and ρ has arity $< \lambda$, then every operation that is $(<\lambda)$-locally in F also preserves ρ.*

All these statements can be easily verified using the definition of 'preserves'.

Conversely, assume that C is a $(<\lambda)$-locally closed clone of $(<\kappa)$-ary operations on A. Our aim is to find a set $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$ of relations such that $C = R^{\perp}$.

For any cardinal $\nu < \kappa$ let $C^{(\nu)}$ denote the set of ν-ary operations in C. We have $C^{(\nu)} \subseteq A^{A^{\nu}}$, so we can consider this set as a $|A^{\nu}|$-ary relation on A. If $X \subseteq A^{\nu}$, then the projection of this relation to X is

$$\pi_X\left(C^{(\nu)}\right) = \{g|_X : g \in C^{(\nu)}\},$$

that is, $\pi_X\left(C^{(\nu)}\right)$ consists of the restrictions of the operations from $C^{(\nu)}$ to the set X. Thus we get the following claim.

Claim 2.3. *A μ-ary operation f preserves the relation $\pi_X\left(C^{(\nu)}\right)$ if and only if for all operations g_m $(m < \mu)$ from $C^{(\nu)}$ the function*

$$f(g_0|_X, g_1|_X, g_2|_X, \dots) = f(g_0, g_1, g_2, \dots)|_X$$

is of the form $g|_X$ for some $g \in C^{(\nu)}$.

Now let $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$ consist of all relations $\pi_X\left(C^{(\nu)}\right)$ where $\nu < \kappa$ is a cardinal and $X \subseteq A^{\nu}$ with $|X| < \lambda$. We will show that $C = R^{\perp}$. The inclusion $C \subseteq R^{\perp}$ is an immediate consequence of Claim 2.3, because C is closed under composition. To verify the reverse inclusion, let f be a $\nu(<\kappa)$-ary operation in $R^{\perp}$. Since C is $(<\lambda)$-locally closed, it is enough to show that f is $(<\lambda)$-locally in C. Let X be any subset of A^{ν} such that $|X| < \lambda$. We have

$$f|_X = f(p_0^{(\nu)}, p_1^{(\nu)}, p_2^{(\nu)}, \dots)|_X = f(p_0^{(\nu)}|_X, p_1^{(\nu)}|_X, p_2^{(\nu)}|_X, \dots) \tag{2.1}$$

where $p_n^{(\nu)}$ $(n < \nu)$ are the ν-ary projection operations. Since C contains the ν-ary projection operations, all $p_n^{(\nu)}|_X$ $(n < \nu)$ belong to $\pi_X\left(C^{(\nu)}\right)$. But f preserves $\pi_X\left(C^{(\nu)}\right) \in R$, so Claim 2.3 combined with (2.1) implies that $f|_X$ belongs to $\pi_X\left(C^{(\nu)}\right)$. Hence there exists an operation $h \in C^{(\nu)}$ such that $f|_X = h|_X$. This proves that f is $(< \lambda)$-locally in C, which completes the proof of the theorem. □

Theorem 2.4. *Let A be a set with at least two elements, and let each one of κ, λ be either an infinite cardinal or ∞. The following conditions on a set $K \subseteq \mathrm{Rel}_A^{(<\lambda)}$ of relations are equivalent:*

(o) *K is Galois closed in the Galois connection $\mathrm{Op}_A^{(<\kappa)} \rightleftharpoons \mathrm{Rel}_A^{(<\lambda)}$;*
(i) *K is a $(< \kappa)$-locally closed relational clone of $(< \lambda)$-ary relations on A;*
(ii) *K is $(< \kappa)$-locally closed and K is closed under definitions by p.p. formulas of length less than*

$$\lambda + \sup\left(\sup_{\mu<\lambda}(|A|^{\mu\nu})^+ : \nu < \kappa,\ |A|^\nu \geq \lambda\right).$$

Proof. Let $K \subseteq \mathrm{Rel}_A^{(<\lambda)}$.

(o) ⇒ (i). If K is Galois closed, then $K = F^\perp$ for some set $F \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations. To see that K is a $(< \kappa)$-locally closed relational clone of $(< \lambda)$-ary relations on A, it suffices to prove the following claim.

Claim 2.5.

(a) *If an operation f preserves some relations ρ_i $(i \in I)$ then f also preserves every relation defined by a p.p. formula using the relations ρ_i $(i \in I)$.*
(c) *If an operation f preserves all members of a set R of relations and f has arity $< \kappa$, then f also preserves every relation that is $(< \kappa)$-locally in R.*

All these statements can be easily verified using the definition of 'preserves'.

(i) ⇒ (ii) is clear.

(ii) ⇒ (o). We need some preparation. Let $\tau = \{t_n : n < \nu\}$ be an arbitrary μ-ary relation $(\mu < \lambda)$. We can arrange the elements of τ in the columns of a $\mu \times \nu$ matrix M, as shown in Figure 1. The rows of M are (not necessarily distinct) elements of A^ν. Let X denote the μ-multiset of the rows of M. Then for any $n < \nu$ we get the entries in the n-th column of M — that is, the coordinates of the n-th element t_n of τ — by projecting the ν-tuples in X onto their n-th coordinates. In other words, by changing the set μ indexing the coordinates of τ to X we get that the element $t_n \colon \mu \to A$ of τ becomes $p_n^{(\nu)}|_X$, the projection operation $p_n^{(\nu)}$ restricted to the μ-multiset X. This proves the following claim.

Claim 2.6. *If $\tau = \{t_n : n < \nu\}$ is a μ-ary relation $(\mu < \lambda)$, then there exists a μ-multiset X in A^ν such that $t_n = p_n^{(\nu)}|_X$ for all $n < \nu$, and hence $\tau = \{p_n^{(\nu)}|_X : n < \nu\}$.*

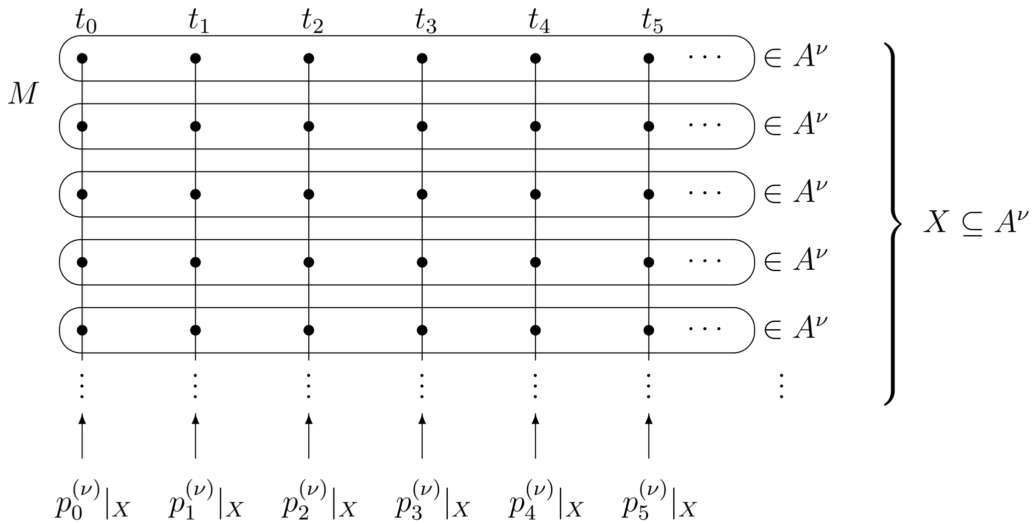

FIGURE 1

To prove the implication (ii) $\Rightarrow$ (o), assume that K satisfies condition (ii). We have to find a set $F \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations such that $K = F^\perp$.

Let $\nu < \kappa$ be a cardinal, and for any ζ-multiset Z in A^ν with $\zeta < \lambda$ consider the set A^Z of all functions $Z \to A$ as a ζ-ary relation. Let R_Z be the collection of all relations $\rho \in K$ such that ρ contains all functions $p_n^{(\nu)}|_Z$ $(n < \nu)$, and let $\gamma_Z = \bigcap R_Z$. For each $\overline{a} \in A^\zeta \setminus \gamma_Z$ choose $\rho_{\overline{a}} \in R_Z$ such that $\overline{a} \notin \rho_{\overline{a}}$. Thus, $\gamma_Z = \bigcap\{\rho_{\overline{a}} : \overline{a} \in A^\zeta \setminus \gamma_Z\}$, so γ_Z can be defined by a (quantifier free) p.p. formula of length $< \omega + (\zeta \cdot |A|^\zeta)^+ = \omega + (|A|^\zeta)^+$ (in ζ free variables). Therefore, assumption (ii) implies that $\gamma_Z \in K$, and γ_Z is the smallest relation in K that contains all functions $p_n^{(\nu)}|_Z$ $(n < \nu)$.

Let $F^{(\nu)}$ be the set of all ν-ary operations f on A such that $f|_Z \in \gamma_Z$ for all ζ-multisets Z in A^ν with $\zeta < \lambda$. Clearly, $F^{(\nu)}$ contains all ν-ary projection operations $p_n^{(\nu)}$ $(n < \nu)$. Furthermore, for any $\mu < \lambda$ and any μ-multiset X in A^ν we have

$$\pi_X\left(F^{(\nu)}\right) = \{f|_X : f \in A^{A^\nu} \text{ and } f|_Z \in \gamma_Z \quad \text{for all } \zeta\text{-multisets } Z \text{ in } A^\nu \text{ with } \zeta < \lambda\}. \tag{2.2}$$

Thus, $\pi_X\left(F^{(\nu)}\right)$ is definable by a p.p. formula in $|A|^\nu$ variables that uses the relations $\gamma_Z \in K$. If $|A|^\nu < \lambda$, then we can choose $X = A^\nu$, and get that $\pi_X\left(F^{(\nu)}\right) = F^{(\nu)} = \gamma_{A^\nu}$ belongs to K and has a p.p. definition by an atomic formula in γ_{A^ν}, which has length $< \lambda$. If $|A|^\nu \geq \lambda$, then for each $\zeta < \lambda$ the number of ζ-multisets Z in A^ν is $|A|^{\nu\zeta}$, so the length of the p.p. formula obtained from this description that defines

$\pi_X\left(F^{(\nu)}\right)$ is either finite (if A, ν, and λ are all finite) or bounded above by

$$\sum_{\zeta<\lambda} \zeta \cdot |A|^{\nu\zeta} = \sum_{\zeta<\lambda} |A|^{\nu\zeta} = \sup_{\zeta<\lambda} |A|^{\nu\zeta},$$

where the last equality holds, because $\sup_{\zeta<\lambda} |A|^{\nu\zeta} \leq \sum_{\zeta<\lambda} |A|^{\nu\zeta} \leq \lambda \sup_{\zeta<\lambda} |A|^{\nu\zeta}$ and $\lambda = \sup_{\zeta<\lambda} \zeta \leq \sup_{\zeta<\lambda} |A|^{\nu\zeta}$. Hence, assumption (ii) yields that $\pi_X\left(F^{(\nu)}\right) \in K$ in this case as well. Since $F^{(\nu)}$ contains the ν-ary projections, $p_n^{(\nu)}|_X$ belongs to $\pi_X\left(F^{(\nu)}\right)$ for all $n < \nu$. But γ_X is the smallest relation in K that contains all functions $p_n^{(\nu)}|_X$ $(n < \nu)$, therefore $\pi_X\left(F^{(\nu)}\right) \supseteq \gamma_X$. The reverse inclusion is clear from (2.2), so we get that

$$\pi_X\left(F^{(\nu)}\right) = \gamma_X \qquad \text{for all } \mu\text{-multisets } X \text{ in } A^\nu \text{ with } \mu < \lambda. \tag{2.3}$$

Now define $F \subseteq \mathrm{Op}_A^{(<\kappa)}$ by $F = \bigcup_{\nu<\kappa} F^{(\nu)}$. We claim that $K = F^\perp$. First we prove the inclusion $K \subseteq F^\perp$. Let $\rho \in K$, say ρ is μ-ary $(\mu < \lambda)$, and let $f \in F^{(\nu)}$ $(\nu < \kappa)$. To prove that $f \perp \rho$, suppose $t_n \in \rho$ for all $n < \nu$. By Claim 2.6 there exists a μ-multiset X in A^ν such that $t_n = p_n^{(\nu)}|_X$ for all $n < \nu$. So, by combining the definition of $F^{(\nu)}$, (2.2), and (2.3) we get that

$$f(t_0, t_1, \ldots) = f(p_0^{(\nu)}|_X, p_0^{(\nu)}|_X, \ldots) = f(p_0^{(\nu)}, p_0^{(\nu)}, \ldots)|_X = f|_X \in \pi_X\left(F^{(\nu)}\right) = \gamma_X.$$

On the other hand, since γ_X is the smallest relation in K that contains all functions $t_n = p_n^{(\nu)}|_X$ $(n < \nu)$, and ρ is another relation with this property, we get that $\gamma_X \subseteq \rho$. Thus, $f(t_0, t_1, \ldots) \in \rho$, proving that $f \perp \rho$.

To prove the reverse inclusion $K \supseteq F^\perp$, let $\rho \in F^\perp$, say ρ is μ-ary $(\mu < \lambda)$. Our goal is to show that $\rho \in K$. Since K is $(< \kappa)$-locally closed, it is enough to show that ρ is $(< \kappa)$-locally in K, that is, for any subset τ of ρ such that $|\tau| < \kappa$ there exists a relation $\sigma \in K$ such that $\tau \subseteq \sigma \subseteq \rho$. Let us fix τ, say $|\tau| = \nu$. By assumption $\nu < \kappa$, and by Claim 2.6 there is a μ-multiset X in A^ν such that $\tau = \{p_n^{(\nu)}|_X : n < \nu\}$. Since ρ is preserved by every operation in F and $\{p_n^{(\nu)}|_X : n < \nu\} \subseteq \rho$, we get $f|_X = f(p_0^{(\nu)}, p_0^{(\nu)}, \ldots)|_X = f(p_0^{(\nu)}|_X, p_0^{(\nu)}|_X, \ldots) \in \rho$ for all $f \in F^{(\nu)}$. Hence, $\pi_X\left(F^{(\nu)}\right) \subseteq \rho$. Combining this with the equality (2.3) we obtain that $\gamma_X \subseteq \rho$. Hence the relation $\sigma = \gamma_X \in K$ satisfies the requirement $\tau \subseteq \sigma \subseteq \rho$.

This completes the proof of Theorem 2.4. $\square$

3. Special cases

Under additional assumptions on the cardinals κ, λ, and $|A|$, we can get simpler descriptions for the Galois closed sets of opererations and relations in the Galois connection $\mathrm{Op}_A^{(<\kappa)} \rightleftharpoons \mathrm{Rel}_A^{(<\lambda)}$. First we state the special cases of Theorems 2.1 and

2.4 for finitary operations and finitary relations on infinite sets. The case when A is finite is treated in Corollary 3.4.

Corollary 3.1. [3, 4, 6] *Let A be an infinite set. In the Galois connection $\mathrm{Op}_A^{(<\omega)} \rightleftharpoons \mathrm{Rel}_A^{(<\omega)}$ between finitary operations and finitary relations on A,*

1. *a set $C \subseteq \mathrm{Op}_A^{(<\omega)}$ of operations is Galois closed if and only if C is a locally closed clone of operations on A; and*
2. *a set $K \subseteq \mathrm{Rel}_A^{(<\omega)}$ of relations is Galois closed if and only if K is locally closed and K is closed under definitions by p.p. formulas of length at most $|A|$.*

Lemma 3.2. *Let A be a set with at least two elements, and let each one of κ, λ be either an infinite cardinal or ∞.*

1. *If $|A|^\nu < \lambda$ for all $\nu < \kappa$, then every subset of $\mathrm{Op}_A^{(<\kappa)}$ is $(<\lambda)$-locally closed.*
2. *If $|A|^\mu < \kappa$ for all $\mu < \lambda$, then every subset of $\mathrm{Rel}_A^{(<\lambda)}$ is $(<\kappa)$-locally closed.*

Proof. (1) Assume that $|A|^\nu < \lambda$ for all $\nu < \kappa$. We have to show that for any set $F \subseteq \mathrm{Op}_A^{(<\kappa)}$, every operation that is $(<\lambda)$-locally in F is in fact in F. Suppose that f is a ν-ary operation ($\nu < \kappa$) and f is $(<\lambda)$-locally in F. By definition this means that for any subset $X \subseteq A^\nu$ with $|X| < \lambda$ there exists a $g \in F$ such that $f|_X = g|_X$. Since $|A|^\nu < \lambda$, this condition has to hold for $X = A^\nu$, so for the corresponding operation g we have $f = g \in F$.

(2) Now assume that $|A|^\mu < \kappa$ for all $\mu < \lambda$. We have to verify that for any set $R \subseteq \mathrm{Rel}_A^{(<\lambda)}$, every relation that is $(<\kappa)$-locally in R is in fact in R. Suppose that ρ is a μ-ary relation ($\mu < \lambda$) and ρ is $(<\kappa)$-locally in R. By definition this means that for any subset $\tau \subseteq \rho$ with $|\tau| < \kappa$ there exists a $\sigma \in R$ such that $\tau \subseteq \sigma \subseteq \rho$. Since $|\rho| \le |A|^\mu < \kappa$, this condition has to hold for $\tau = \rho$, so for the corresponding relation σ we have $\rho = \sigma \in R$. $\square$

Theorem 3.3. *Let A be a set, and let κ be either a strong limit cardinal or ∞. If $1 < |A| < \kappa$, then in the Galois connection $\mathrm{Op}_A^{(<\kappa)} \rightleftharpoons \mathrm{Rel}_A^{(<\kappa)}$*

1. *a set $C \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations is Galois closed if and only if C is a clone of $(<\kappa)$-ary operations on A;*
2. *a set $K \subseteq \mathrm{Rel}_A^{(<\kappa)}$ of relations is Galois closed if and only if*
   (i) *K is a relational clone of $(<\kappa)$-ary relations on A, or equivalently,*
   (ii) *K is closed under definitions by p.p. formulas of length $< \kappa$.*

Proof. If $1 < |A| < \kappa$ and κ is either a strong limit cardinal or $\kappa = \infty$, then $|A|^\nu < \kappa$ for all $\nu < \kappa$. Therefore when we apply Theorems 2.1 and 2.4 for the case $\lambda = \kappa$, we see that the requirements on $(<\kappa)$-local closure can be omitted (by Lemma 3.2), and the strict upper bound in Theorem 2.4(ii) for the length of p.p. formulas is κ. $\square$

The corollaries below state the two most important special cases of Theorem 3.3: in the first case $\kappa = \omega$ and A is finite, while in the second case $\kappa = \infty$ and A is arbitrary.

Corollary 3.4. [1, 2, 5] *Let A be a finite set with at least two elements. In the Galois connection $\mathrm{Op}_A^{(<\omega)} \rightleftharpoons \mathrm{Rel}_A^{(<\omega)}$ between finitary operations and finitary relations on A,*

(1) *a set $C \subseteq \mathrm{Op}_A^{(<\omega)}$ of operations is Galois closed if and only if C is a clone of operations on A; and*
(2) *a set $K \subseteq \mathrm{Rel}_A^{(<\omega)}$ of relations is Galois closed if and only if K is closed under definitions by p.p. formulas of finite length.*

Corollary 3.5. *Let A be a set with at least two elements. In the Galois connection $\mathrm{Op}_A^{(<\infty)} \rightleftharpoons \mathrm{Rel}_A^{(<\infty)}$ between operations and relations of arbitrary arity on A,*

(1) *a set $C \subseteq \mathrm{Op}_A^{(<\infty)}$ of operations is Galois closed if and only if C is a clone of $(<\infty)$-ary operations on A; and*
(2) *a set $K \subseteq \mathrm{Rel}_A^{(<\infty)}$ of relations is Galois closed if and only if K is a relational clone of $(<\infty)$-ary relations on A.*

Theorem 3.6. *Let A be a set with at least two elements, and let each one of κ, λ be either an infinite cardinal or ∞. If $|A|^\nu < \lambda$ for all $\nu < \kappa$, then in the Galois connection $\mathrm{Op}_A^{(<\kappa)} \rightleftharpoons \mathrm{Rel}_A^{(<\lambda)}$*

(1) *a set $C \subseteq \mathrm{Op}_A^{(<\kappa)}$ of operations is Galois closed if and only if C is a clone of $(<\kappa)$-ary operations on A.*
(2) *a set $K \subseteq \mathrm{Rel}_A^{(<\lambda)}$ of relations is Galois closed if and only if K is $(<\kappa)$-locally closed and K is closed under definitions by p.p. formulas of length $<\lambda$.*

Proof. By our assumption that $|A|^\nu < \lambda$ for all $\nu < \kappa$, statement (1) follows immediately from Theorem 2.1 and Lemma 3.2 (1). The same assumption also yields that the strict upper bound in Theorem 2.4(ii) for the lengths of p.p. formulas is λ. Therefore, statement (2) follows from Theorem 2.4. □

The next corollary states the most frequently used special case of Theorem 3.6, namely the case when $\kappa = \omega$ and $\lambda = \infty$.

Corollary 3.7. *Let A be a set with at least two elements. In the Galois connection $\mathrm{Op}_A^{(<\omega)} \rightleftharpoons \mathrm{Rel}_A^{(<\infty)}$ between finitary operations and relations of arbitrary arity on A,*

(1) *a set C of finitary operations on A is Galois closed if and only if C is a clone on A.*
(2) *a set $K \subseteq \mathrm{Rel}_A^{(<\infty)}$ of relations is Galois closed if and only if K is a locally closed relational clone of $(<\infty)$-ary relations on A.*

4. An alternative characterization of local closure for relations

A set R of relations of the same arity will be called $(<\kappa)$**-directed** if for every subset $S \subseteq R$ with $|S| < \kappa$ there exists a relation $\rho \in R$ such that $\sigma \subseteq \rho$ for all $\sigma \in S$. We will say that a set K of relations is **closed under** $(<\kappa)$**-directed union** if for any $(<\kappa)$-directed subset R of K, the union $\bigcup R$ of the relations in R belongs to K. For $\kappa = \omega$ these notions specialize to the usual notions of a **directed** set of relations, and closure under **directed union**.

Proposition 4.1. *Let A be a set with at least two elements, let λ be either an infinite cardinal or ∞, and let $K \subseteq \mathrm{Rel}_A^{(<\lambda)}$ be a set of relations that is closed under intersection. For any regular cardinal κ the following conditions are equivalent:*

(i) *K is $(<\kappa)$-locally closed;*
(ii) *K is closed under $(<\kappa)$-directed union.*

Proof. (i)⇒(ii). Assume first that K is $(<\kappa)$-locally closed, and let R be a $(<\kappa)$-directed set of relations in K. To prove the implication (i)⇒(ii) it suffices to show that the relation $\bigcup R$ is $(<\kappa)$-locally in K. Let $\tau \subseteq \bigcup R$ be such that $|\tau| < \kappa$. For any $t \in \tau$ there exists a relation $\sigma_t \in R$ with $t \in \sigma_t$. But R is $(<\kappa)$-directed, therefore R contains a relation ρ such that $\sigma_t \subseteq \rho$ for all $t \in \tau$. Thus $\rho \in K$ and $\tau \subseteq \rho \subseteq \bigcup R$. Hence $\bigcup R$ is $(<\kappa)$-locally in K. This completes the proof of the implication (i)⇒(ii).

(ii)⇒(i). Assume that K is closed under $(<\kappa)$-directed union, and let ρ be a relation that is $(<\kappa)$-locally in K. Our goal is to show that $\rho \in K$. The assumption that ρ is $(<\kappa)$-locally in K means that for any subset $\tau \subseteq \rho$ with $|\tau| < \kappa$ there exists a relation $\sigma \in K$ such that $\tau \subseteq \sigma \subseteq \rho$. Since K is closed under intersection, it contains a smallest relation σ with $\tau \subseteq \sigma$; this relation will be denoted by $\langle \tau \rangle$. Thus $\tau \subseteq \langle \tau \rangle \subseteq \rho$. Let R denote the set of all relations $\langle \tau \rangle \in K$ such that $\tau \subseteq \rho$ and $|\tau| < \kappa$. Then

$$\rho = \bigcup(\tau : \tau \subseteq \rho,\ |\tau| < \kappa) \subseteq \bigcup(\langle \tau \rangle : \tau \subseteq \rho,\ |\tau| < \kappa) = \bigcup R \subseteq \rho,$$

so $\rho = \bigcup R$. We show that if κ is a regular cardinal, then R is a $(<\kappa)$-directed set. Let $\{\sigma_n : n < \nu\}$ be a subset of R with $\nu < \kappa$. Every σ_n has the form $\langle \tau_n \rangle$ for some $\tau_n \subseteq \rho$ with $|\tau_n| < \kappa$. Since κ is regular, therefore $\sum_{n<\nu} |\tau_n| < \kappa$. So $\tau = \bigcup_{n<\nu} \tau_n$ has cardinality $< \kappa$. Thus $\langle \tau \rangle \in R$ and $\sigma_n = \langle \tau_n \rangle \subseteq \langle \tau \rangle$ for all $n < \nu$. This shows that R is $(<\kappa)$-directed. Since K is closed under $(<\kappa)$-directed union, we conclude that $\rho = \bigcup R \in K$, which completes the proof of the lemma. □

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