ANALYSIS 1
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. State the theorems. Include all necessary hypotheses.

(a) The Monotone Convergence Theorem.

If a sequence is monotone and bounded, then it converges.

(b) The Bolzano-Weierstrass Theorem.

Every bounded sequence contains a convergent subsequence.

2. Define the given word or phrase.

(a) supremum (of a set of real numbers).

The supremum of a set $A \subseteq \mathbb{R}$ is a number $s \in \mathbb{R}$ such that

(i) $\forall a \in A(a \leq s)$, which says that s is an upper bound for A, and

(ii) $(\forall a \in A(a \leq t)) \to (s \leq t))$, which says that s is less or equal to any upper bound t for A.

(b) convergent sequence.

$(a_i)_{i \in \mathbb{N}^*}$ is a convergent sequence if

$$(\exists L)(\forall \varepsilon > 0)(\exists N)(\forall i)((i > N) \to (|a_i - L| < \varepsilon)).$$

3. Write formal sentences:

(a) Write a sentence that is meaningful for any ordered field, and is true in $\mathbb{Q}$ but false in $\mathbb{R}$.

One could write a sentence saying that $x^2 = 2$ has no solution, for example
$$\neg \exists x \ (x^2 = 2).$$
If you prefer sentences with all quantifiers in front, then we could write this as
$$\forall x \ (x^2 \neq 2).$$

(b) Write a formal sentence that expresses that the sequence $(a_i)_{i \in \mathbb{N}^*}$ diverges.

$$(\forall L)(\exists \varepsilon > 0)(\forall N)(\exists i)((i > N) \wedge (|a_i - L| \not< \varepsilon))$$

4. Determine the truth value, and justify your answer by giving an appropriate strategy.

(a) Is $\forall x \ \exists y \ (x = y)$ true in $\mathbb{R}$?

The sentence is true. I give a winning strategy for $\exists$.
$\forall$: Chooses some x. We can't control this choice.
$\exists$: Choose $y = x$.

(b) Is
$$(\forall L)(\exists \varepsilon > 0)(\forall N)(\exists i)((i > N) \wedge (|1 - L| \not< \varepsilon))$$
true in $\mathbb{R}$? (In other words, is $(1, 1, 1, \ldots)$ divergent?)

The sentence is false. I give a winning strategy for $\forall$.
$\forall$: Choose $L = 1$.
$\exists$: Chooses some ε. We can't control this choice, but no harm assuming that $\exists$ chose ε satisfying the condition $\varepsilon > 0$.
$\forall$: Choose $N = 1$
$\exists$: Chooses some i. We can't control this choice, but no harm assuming that $\exists$ chose i satisfying the condition $i > N$.