Solutions to HW 9.

1. (Exercise 5.2.2.) Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, lets assume the functions are defined on all of $\mathbb{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

Let f be the Dirichlet function and let $g = 1 - f$. Neither function is continuous anywhere, so neither is differentiable anywhere. But $f(x)g(x) = 0$ everywhere, hence fg is differentiable everywhere.

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

Let f be the Dirichlet function and let g be the zero function.

(c) A function f not differentiable at zero and a function g differentiable at zero where $f + g$ is differentiable at zero.

Can't happen. If g and $f + g$ are differentiable, then $f'(0) = ((f + g) - g)'(0) = (f + g)'(0) - g'(0)$, which exists.

(d) A function f differentiable at zero but not differentiable at any other point.

Let $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ for $x \neq 0$.

2. (Exercise 5.2.9.) Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

TRUE. Since f' is nonconstant, there exist $a < b$ such that $f'(a) \neq f'(b)$. Pick any irrational number α between $f'(a)$ and $f'(b)$ (which is possible, since the irrationals are dense in $\mathbb{R}$), and use Darboux's Theorem to produce a c between a and b such that $f'(c) = \alpha$.

(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

FALSE. Recall that the function defined by $g(0) = 0$ and $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ is differentiable everywhere, that $g'(0) = 0$, and that in any neighborhood of 0 the function g oscillates between values close to -1 and $+1$.

So define a function $f(x) = x/2 + g(x)$. That is, $f(0) = 0$ and $f(x) = x/2 + x^2 \sin(1/x)$ for $x \neq 0$. Then $f'(0) = 1/2 > 0$ and $f'(x) = 1/2 + 2x \sin(1/x) - \cos(1/x)$ if $x \neq 0$. Here $f'(0) > 0$, but in every neighborhood of 0 the function $f'(x)$ oscillates between values close to $-1/2$ and $+3/2$. There is no open neighborhood of 0 on which f' is always positive.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

TRUE. To prove this by contradiction, assume that $f'(0) = K \neq L$. As a first case, assume that $K > L$. (The case $K < L$ is similar.) Let $\varepsilon = (K - L)/2$. Choose $\delta > 0$ so that $0 < |x - 0| < \delta$ implies $|f'(x) - L| < \varepsilon$. Then for $x \in (-\delta, \delta)$ we have $f'(x) \in (L - \varepsilon, L + \varepsilon)$ except at $x = 0$ where we have $f'(0) = K$. Now $L + \varepsilon = (K + L)/2 < K$, so there exists u such that $L + \varepsilon < u < K$. Since $\delta/2 \in (-\delta, \delta)$ and $\delta/2 \neq 0$ we have

$$f'(\delta/2) < L + \varepsilon < u < K = f'(0),$$

so by Darboux's Theorem we should have some $c \in (0, \delta/2)$ such that $f'(c) = u$, but this is not so since $f'((0, \delta/2)) \subseteq (L - \varepsilon, L + \varepsilon)$ and the latter interval does not contain u. This contradiction completes the proof.

3. (Exercise 5.3.2.) Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

If f fails to be 1-1 on A, then there exist $a, b \in A$, $a < b$, such that $f(a) = f(b)$. By the MVT there is some $c \in (a, b)$ such that

$$0 = \frac{f(b) - f(a)}{b - a} = f'(c) \neq 0,$$

which is a contradiction. Hence f is 1-1 on A.

To show that converse is false, consider $f(x) = x^3$ on $A = [-1, 1]$. The function f is 1-1 on A, but it is not true that $f'(x) \neq 0$ on A, since $f'(0) = 0$.