Solutions to HW 8.

1. (Exercise 4.5.3.) A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a,b]$.

We must give a winning strategy for $\exists$ in the game

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(|x-c| < \delta \to |f(x) - f(c)| < \varepsilon)$$

for $c \in [a,b]$. We argue this for $a < c < b$ only, although the argument can be modified for $a = c$ or $c = b$.

- Let $\forall$ choose $\varepsilon > 0$.
- If $f(a) = f(c)$, then f is constant on $[a,c]$ since f is monotone increasing. If $f(a) < f(c)$, then choose u_1 so that it is between $f(a)$ and $f(c)$ and also between $f(c) - \varepsilon$ and $f(c)$. In either case, by the constancy of f on $[a,c]$ or by the IVP, there exists some $d_1 < c$ such that $f(d_1) = u_1$. Similarly, we can choose some u_2 so that it is between $f(c)$ and $f(b)$ and also between $f(c)$ and $f(c)+\varepsilon$. As before, we can find some $d_2 > c$ such that $f(d_2) = u_u$. Now let $\exists$ choose $\delta > 0$ so that $(c-\delta, c+\delta) \subseteq (d_1, d_2)$.

  By these choices, and the monotonicity of f,

  $$f(c) - \varepsilon < f(d_1) \leq f((c-\delta, c+\delta)) \leq f(d_2) < f(c) + \varepsilon.$$

- Since $f((c-\delta, c+\delta)) \subseteq (f(c) - \varepsilon, f(c) + \varepsilon)$, whatever $\forall$ chooses for x we will satisfy the condition "$|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$".

2. (Exercise 4.5.4.) Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x \in A \mid f(x) = f(y) \text{ for some } y \neq x \text{ and } y \in A\}.$$

Show F is either empty or uncountable.

We must show that if $F \neq \emptyset$, then F is uncountable. Assume that $F \neq \emptyset$, so there is some $a < b$ such that $g(a) = g(b)$. I will use the fact that any interval with more than one point is uncountable.

If g is constant on $[a,b]$, then $[a,b] \subseteq F$, which implies that F is uncountable.

If $\exists c \in [a,b]$ such that $g(a) \neq g(c)$, then by the IVT for every u strictly between $g(a)$ and $g(c)$ there is a $d_u \in (a,c)$ such that $g(d_u) = u$ and an $e_u \in (c,b)$ such that $g(e_u) = u$. Since $d_u \neq e_u$, but $g(d_u) = g(e_u)$ we have $d_u, e_u \in F$. The map $u \mapsto d_u$ is a 1-1 function from the interval between $g(a)$ and $g(c)$ into F, which forces F to be uncountable.

3. (Exercise 4.5.7.) Let f be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0,1]$.

If $f(0) = 0$, then 0 is a fixed point of f, and we are done (we found a fixed point). If $f(1) = 1$, then 1 is a fixed point of f, and we are done.

If we are not yet done, then $f(0) \in (0, 1]$, so $f(0) > 0$, and $f(1) \in [0, 1)$, so $f(1) < 1$. Hence the continuous function $g(x) = x - f(x)$ satisfies $g(0) < 0 < g(1)$. By the Intermediate Value Theorem, there is some $c \in (0, 1)$ such that $0 = g(c) = c - f(c)$, hence $f(c) = c$. In this case, c is a fixed point of f and we are done.