Solutions to HW 7.

1. (Exercise 4.3.6(a)(b)(c).) Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0.

Let f be the Dirichlet function, namely

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else.} \end{cases}$$

Let $g(x) = 1 - f(x)$, that is

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{else.} \end{cases}$$

Neither f nor g is continuous anywhere, but $f(x)+g(x) = 1$ is a continuous (constant) function, and $f(x)g(x) = 0$ is a continuous (constant) function.

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

This can't happen. If $f(x) + g(x)$ and $f(x)$ are continuous at 0, then $(f(x) + g(x)) - f(x) = g(x)$ is continuous at 0.

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

Let $f(x) = 0$ be the constant zero function and let g be any discontinuous function (like the Dirichlet function). Then $f(x)g(x) = 0$ is constant zero, hence continuous.

2. Show that a nonempty subset $C \subseteq \mathbb{R}$ is closed iff there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $C = g^{-1}(0)$. (Hint for the proof of $\Leftarrow$: explain why the inverse image of a closed set is closed. Hint for the proof of $\Rightarrow$: you may cite parts of Exercise 4.3.12 if needed.)

Claim. A function g has the property that "O open implies $g^{-1}(O)$ open" if and only if g has the property that "C closed implies $g^{-1}(C)$ closed".

Proof of Claim. Assume that g has the property that "O open implies $g^{-1}(O)$ open". Now let C be closed. We must show that $g^{-1}(C)$ is closed.

Since C is closed, $\mathbb{R} \setminus C$ is open. By the property of g, $g^{-1}(\mathbb{R} \setminus C)$ is open. Now an element x belongs to this open set iff

$$\begin{aligned} x \in g^{-1}(\mathbb{R} \setminus C) \quad &\text{iff} \quad g(x) \in \mathbb{R} \setminus C \\ &\text{iff} \quad g(x) \notin C \\ &\text{iff} \quad x \notin g^{-1}(C) \\ &\text{iff} \quad x \in \mathbb{R} \setminus g^{-1}(C). \end{aligned}$$

Hence the open set $g^{-1}(\mathbb{R} \setminus C)$ equals $\mathbb{R} \setminus g^{-1}(C)$. But if this latter set is open, then $g^{-1}(C)$ must be closed.

Altogether this shows that if "O open implies $g^{-1}(O)$ open", then also "C closed implies $g^{-1}(C)$ closed". (The same argument proves the reverse implication.) End of Proof of Claim.

It follows from the Claim, and from the fact that singleton sets in $\mathbb{R}$ are closed, that if $g \colon \mathbb{R} \to \mathbb{R}$ is continuous, then $g^{-1}(0)$ is closed.

For the converse, assume that $C \subseteq \mathbb{R}$ is closed. It is shown in Exercise 4.3.12 that

$$g(x) = \inf\{|x - c| \mid c \in C\}$$

is a continuous function that does not vanish off of C. (That is, $d \notin C$ implies that $g(d) > 0$.) It is easy to see that g does vanish on C, that is $g(b) = 0$ for any $b \in C$. To see why, note that if $b \in C$, then $g(b)$ is the infimum of the set $\{|b - c| \mid c \in C\}$. But the elements of this set are nonnegative, and one of them is zero, so the infimum is zero.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there is a number p such that $f(x+p) = f(x)$ for every x. (For example, $\sin(x)$ is periodic with $p = 2\pi$, since $\sin(x + 2\pi) = \sin(x)$.) Prove that a continuous periodic function is uniformly continuous.

Let $p > 0$ be a period of f. The function f is continuous on all of $\mathbb{R}$, hence continuous on $[0, 2p]$, hence uniformly continuous on $[0, 2p]$ by the Heine-Cantor Theorem. This means that $\forall \varepsilon > 0$ there exists $\delta > 0$ depending on ε only (which I emphasize by writing $\delta = \delta_\varepsilon$) such that for all $L, x \in [0, 2p]$ we have

$$|x - L| < \delta_\varepsilon \to |f(x) - f(L)| < \varepsilon.$$

For any positive choice of δ_ε that makes this statement true, any smaller positive choice will make the statement true, so by shrinking δ_ε if necessary we may assume that $\delta_\varepsilon < p/2$ for any ε.

Now, we must show that f is uniformly continuous on all of $\mathbb{R}$. Given any $\varepsilon > 0$, choose the same value of δ that worked on the interval $[0, 2p]$, namely $\delta = \delta_\varepsilon$. For any $L \in \mathbb{R}$ there is some integer multiple of the period, kp, so that $L + kp \in [p/2, 3p/2]$. That is, $L + kp$ is a translate of L by an integer number of periods such that it lies in $[0, 2p]$ and is at least $p/2$ from the endpoints.

Now, for any x, we have $|x - L| = |(x + kp) - (L + kp)|$, and we have $|f(x) - f(L)| = |f(x + kp) - f(L + kp)|$ since f has period p. Hence, replacing the appropriate subformulas of

$$|(x + kp) - (L + kp)| < \delta_\varepsilon \to |f(x + kp) - f(L + kp)| < \varepsilon,$$

which holds because $x + kp, L + kp \in [0, 2p]$, we get

$$|x - L| < \delta_\varepsilon \to |f(x) - f(L)| < \varepsilon,$$

which was what we needed to show to prove that f is uniformly continuous.