Solutions to HW 6.

1. (Exercise 3.3.4.) Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$.

Both. ($K \cap F$ is both closed and bounded, so closed and compact.)

(b) $\overline{F^c \cup K^c}$.

Closed, but may not be compact. (If $K = F = \emptyset$, then $\overline{F^c \cup K^c} = \mathbb{R}$.)

(c) $K \setminus F = \{x \in K \mid x \notin F\}$.

May be neither. (For example, let $K = [0,1]$ and $F = \{0\}$. Then $K \setminus F = (0,1]$, which is neither closed nor compact.)

(d) $\overline{K \cap F^c}$.

Both. ($\overline{K \cap F^c}$ is closed by construction. It is contained in K, so it is also compact.)

2. (Exercise 3.3.6.) Verify that the following three statements are true if every blank is filled in with the word "finite." Which are true if every blank is filled in with the word "compact." Which are true if every blank is filled in with the word "closed."

I think the author meant all sets to be nonempty in this problem, especially part (a).

I explain why all parts are true for "nonempty+finite", I assert without proof that they are all true for "nonempty+compact", and I give an example to show they are all false for "closed".

(a) We can prove that a nonempty finite set $A = \{a_1 \ldots, a_k\}$ has a maximum by induction. If $A = \{a_1\}$, then $\max\{A\} = a_1$. Now suppose $A = \{a_1, \ldots, a_k, a_{k+1}\}$ is given. By induction, $\{a_1, \ldots, a_k\}$ has a maximum element, say a_i. Then $\max\{a_i, a_{k+1}\}$ is the maximum of A. (If A is empty, then it has no maximum element because it has no element.)

Assertion: Every nonempty compact set A has a maximum. (If A is empty, then A is compact, but it has no maximum element.)

Not true for closed. Example: $A = \mathbb{R}$ has no maximum element.

(b) If $|A| = m$ and $|B| - n$, then $|A + B| \leq m + n$, so if A and B are finite, then $A + B = \{a + b \mid a \in A, b \in B\}$ is also finite.

Assertion: If A and B are compact, then $A + B = \{a + b \mid a \in A, b \in B\}$ is also compact.

Not true for closed. Example: $A = \mathbb{Z}$, $B = \{2+\frac{1}{2}, 3+\frac{1}{3}, 4+\frac{1}{4}, \ldots\}$. $A+B$ contains no integers, but does contain $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Thus $0 \in \overline{(A + B)} \setminus (A + B)$.

(c) If $\{A_n \mid n \in \mathbb{N}\}$ are finite, and every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well. To see this, consider the sets

$$A_1 \supseteq A_1 \cap A_2 \supseteq A_1 \cap A_2 \cap A_3 \supseteq \dots.$$

These are finite intersections, so they are all nonempty. This is a descending chain of subsets of the finite set A_1. A_1 has only finitely many subsets, so for some N we have

$$\bigcap_{i=1}^{N} A_i = \bigcap_{i=1}^{N+1} A_i = \cdots = B \neq \emptyset.$$

But then $\bigcap_{i=1}^{\infty} A_i = B \neq \emptyset$ as well.

Assertion: If $\{A_n \mid n \in \mathbb{N}\}$ is a collection of compact sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

Not true for closed. Example: Let $A_n = [n, \infty)$.

3. For each part below, give an example of a subset $A \subseteq \mathbb{R}$ or $A \subseteq \mathbb{R}^2$ such that

(a) A is connected, but A° and ∂A are not connected.

Let B be the punctured disk consisting all points on and inside the circle $\gamma_1 : x^2 + y^2 = 1$ *except* the point $(0, 0)$. Let C be the tangent disk consisting of all points on and inside the circle $\gamma_2 : \ (x - 2)^2 + y^2 = 1$. Each of B and C are connected, and they overlap at the single point $(x, y) = (1, 0)$, so the set $A = B \cup C$ is also connected.

But A° is not connected, since $\{B^\circ, C^\circ\}$ is a separation of A°. Also, ∂A is not connected, since $\{U, V\}$ is a separation for $U = \{(0, 0)\}$ and $V = \gamma_1 \cup \gamma_2$.

(b) A is not connected, but A° and ∂A are connected.

Let $A = \mathbb{Q} \subseteq \mathbb{R}$. A is disconnected because the only connected subsets of $\mathbb{R}$ are the intervals. But $A^\circ = \emptyset$, which is connected, and $\partial A = \mathbb{R}$, which is connected,