Solutions to HW 4.

1. (Exercise 2.2.4.) Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

$$(0, 1, 0, 1, 0, 1, \ldots)$$

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

Such a sequence cannot exist.

Assume that $(a_i)_{i\in\mathbb{N}^*}$ has infinitely many 1's, and converges to $L \neq 1$. Let $\varepsilon = |1-L|/2$. Choose N so that whenever $i > N$ it is the case that $|a_i - L| < \varepsilon$. Choose $i > N$ so that $a_i = 1$. Now $|a_i - L| = |1-L| < \varepsilon = |1-L|/2$. But $|1-L| < |1-L|/2$ leads to $|1-L| < 0$, which is impossible.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

$$(a_1, a_2, a_3, a_4, a_5, \ldots) = (0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, \ldots)$$

Here $a_n = 0$ if n is a power of 2, and $a_n = 1$ otherwise.

2. (Exercise 2.3.7 (a), (b), (c).) Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges.

Let $x_n = n$ and $y_n = -n$. Then

$$(x_1) = (1, 2, 3, \ldots) \qquad \text{and} \qquad (y_n) = (-1, -2, -3, \ldots)$$

both diverge, but their sum $(x_n + y_n) = (0, 0, 0, \ldots)$ converges.

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges.

Such sequences cannot exist.

Assume that (x_n) and $(x_n + y_n)$ converge. If $a_n = -x_n$ and $b_n = x_n + y_n$, then both (a_n) and (b_n) converge, by assumption and by the Algebraic Limit Theorem. Hence $(a_n + b_n) = (y_n)$ converges, by the Algebraic Limit Theorem.

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges.

$$(b_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

3. (Exercise 2.4.4(a).) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbb{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

Proof. This is a proof by contradiction, so assume that $\mathbb{R}$ is not Archimedean.

Since $\mathbb{R}$ is not Archimedean, $\mathbb{N}$ is bounded above, which implies that $(a_n) := (1, 2, 3, \ldots)$ is a monotone increasing bounded sequence. By the Monotone Convergence Theorem, this sequence has a limit, $L \in \mathbb{R}$. There must exist some N such that for $i > N$ we have $a_i \in (L-1, L+1)$, or equivalently $L - 1 < a_i < L + 1$. In particular, this means that

$$L - 1 < a_{N+1} < a_{N+2} < a_{N+3} < \cdots < L + 1.$$

But if $L - 1 < a_{N+1} < L + 1$, then (by adding 2 throughout) we have $L + 1 < a_{N+3} < L + 3$, leading to the contradiction that $a_{N+3} < L + 1 < a_{N+3}$.