Solutions to HW 3.

1. Does the non-Archimedean field $\mathbb{R}(t)$ satisfy the Nested Interval Property? Explain.

No. To see this, let's argue that the nested intersection $\bigcap_{n=1}^{\infty}[n, t/n]$ is empty.

This is a proof by contradiction. Assume that the rational function $P(t)/Q(t)$ of $\mathbb{R}(t)$ belongs to the intersection. Here $P(t) = p_k t^k + \cdots + p_1 + p_0$ and $Q(t) = q_\ell t^\ell + \cdots + q_1 + q_0$ are polynomials with real coefficients, both must be nonzero, and both may be assumed to have positive leading coefficients (since this is true of every element of the first interval, $[1, t]$). Here the 'leading coefficient' of P is $p_k > 0$ and the 'leading coefficient' of Q is $q_\ell > 0$. Also, the 'degree' of P is $\deg(P) = k$, while the 'degree' of Q is $\deg(Q) = \ell$.

Lemma 1. *In the ordered field $\mathbb{R}(t)$, if $A(t) = a_k t^k + \cdots + a_1 + a_0$ and $B(t) = b_\ell t^\ell + \cdots + b_1 + b_0$ both have positive leading coefficients (i.e., $a_k, b_\ell > 0$), and $0 < A - nB$ for all $n \in \mathbb{N}$, then $\deg(A) > \deg(B)$.*

Proof. There are cases to consider:

(1) ($\deg(A) < \deg(B)$) The leading coefficient of $A - nB$ is $-nb_\ell < 0$, a contradiction to $0 < A - nB$.

(2) ($\deg(A) = \deg(B)$) Then $k = \ell$ and the leading coefficient of $A - nB$ is $a_k - nb_k$. If this is positive for all n, then $a_k > nb_k$ for all n, or $a_k/b_k > n$ for all n. But this contradicts the Archimedean property of $\mathbb{R}$: the real number a_k/b_k would be larger than any natural number.

(3) ($\deg(A) > \deg(B)$) This is the only remaining case, so it must hold.

Overall, our conclusion is that $\deg(A) > \deg(B)$. □

We have assumed that P/Q belongs to the intersection $\bigcap_{n=1}^{\infty}[n, t/n]$, so $n < P/Q < t/n$ for all n. The left hand inequalities, $n < P/Q$ for all n, taken together, are equivalent to the statement that $0 < P - nQ$ holds for all n. By the lemma, $\deg(P) > \deg(Q)$. The right hand inequalities, $P/Q < t/n$ for all n, are equivalent to the statement that $0 < tQ - nP$ holds for all n. By the lemma, $\deg(tQ) > \deg(P)$.

Now we have $\deg(P) < \deg(tQ) = 1 + \deg(Q) < 1 + \deg(P)$. That is, the positive integer $\deg(tQ)$ lies strictly between the consecutive positive integers $\deg(P)$ and $1 + \deg(P)$, which is impossible. This is the contradiction that completes the proof.

{Comments on this solution: How does one know (or decide) to include a lemma like the one above? Answer: If P/Q belongs to $[n, t/n]$ for all n, then $n < P/Q$ for all n and $P/Q < t/n$ for all n. One extracts conclusions from these assumptions by the same arguments. Typically you don't realize this until you have written both arguments out. But then, rather than write the argument twice, you should write it once as a lemma and refer to the lemma twice.}

2. Show that if $S \subseteq [0,1]$ is uncountable, then there is a real number $r \in [0,1]$ such that both $[0,r] \cap S$ and $[r,1] \cap S$ are uncountable.

Define

$$A = \{r \in [0,1] \mid [0,r] \cap S \text{ is countable}\},$$

and

$$B = \{r \in [0,1] \mid [r,1] \cap S \text{ is countable}\}.$$

The set A is nonempty ($0 \in A$) and bounded above by 1, so $\sup(A)$ exists. Similarly $\inf(B)$ exists.

Claim 2. $\sup(A) < \inf(B)$.

Proof of Claim. If the claim is not true, then $\inf(B) \leq \sup(A)$, so there is a real number t such that $\inf(B) \leq t \leq \sup(A)$. For each n we have that $t - (1/n) < \sup(A)$, so by Lemma 1.3.8 there is a number $a_n \in A \cap (t - (1/n), 1]$. Similarly, there is a number $b_n \in B \cap [0, t + (1/n))$. This implies that $S \cap [0, t - (1/n)]$ and $S \cap [t + (1/n), 1]$ are both countable. Since a countable union of countable sets is countable,

$$\begin{aligned} S &= S \cap [0,1] \\ &= S \cap \left(\bigcup_{n=1}^{\infty} [0, t - 1/n] \ \cup \bigcup_{n=1}^{\infty} [t + (1/n), 1] \ \cup \{t\}\right) \\ &= \left(\bigcup_{n=1}^{\infty} S \cap [0, t - (1/n)]\right) \cup \left(\bigcup_{n=1}^{\infty} S \cap [t + (1/n), 1]\right) \cup (S \cap \{t\}) \end{aligned}$$

is countable. This contradicts the uncountability of S. □

To complete the solution, the Claim shows that $\sup(A) < \inf(B)$, so there is some real number r such that $0 \leq \sup(A) < r < \inf(B) \leq 1$. For this r we have that $S \cap [0,r]$ uncountable, else $r \in A$ and we get the contradiction that $\sup(A) < r$ yet $r \in A$. In a similar way, using B, we get that $S \cap [r,1]$ is uncountable.

3. (Exercise 1.5.8.) Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countably infinite.

Let $B_k = B \cap \left[\frac{1}{k}, \infty\right)$. Any sum of more than $2k$ members of B_k will exceed $2k \cdot \frac{1}{k} = 2$, so it must be that $|B_k| \leq 2k$. By the Archimedean property of $\mathbb{R}$, $B = \bigcup_{i=1}^{\infty} B_k$. Hence B is a countable union of countable sets, making B countable. (Countable = finite or countably infinite.)