Solutions to HW 2.

1. (Exercise 1.3.3.)

   (a) Let A be nonempty and bounded below, and define

   $$B = \{b \in \mathbb{R} \mid b \text{ is a lower bound for } A\}.$$

   Show that $\sup(B) = \inf(A)$.

   Since A is bounded below, $B \neq \emptyset$. There exists at least one $a \in A$, and any such a is an upper bound for B. By the Completeness Axiom, $\sup(B)$ exists, and necessarily $\sup(B) \leq a$ for this a. Since $a \in A$ is arbitrarily chosen, we get that $\sup(B) \leq a$ for all $a \in A$, hence $\sup(B)$ is a lower bound for A.
   Now we argue that $\sup(B)$ is the greatest lower bound for A, that is that $\sup(B) = \inf(A)$. If this is not the case, then there is some $s > \sup(B)$ that is a lower bound for A. By the definition of B, $s \in B$, so $s \leq \sup(B)$. Altogether we get the contradiction that $\sup(B) < s \leq \sup(B)$, so it must be the case that $\sup(B)$ is the greatest lower bound for A.

   (b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

   Part (a) shows that if A is a nonempty subset of $\mathbb{R}$ that is bounded below, then $\inf(A)$ exists and equals $\sup(B)$. Therefore we don't need another axiom to prove $\inf(A)$ exists.

2. (Exercise 1.3.6. (a), (b), (c).) Given sets A and B, define

   $$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

   Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup(A) + \sup(B)$.

   (a) Let $s = \sup(A)$ and $t = \sup(B)$. Show $s + t$ is an upper bound for $A + B$.

   Choose an element $a + b \in A + B$. Since $a \leq s$ and $b \leq t$, we have $a + b \leq a + t \leq s + t$, or $a + b \leq s + t$. This proves that $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

For any $b \in B$, the element $a + b \leq u$, since u is an upper bound for $A + B$. Thus $b \leq u - a$ for any $b \in B$ and for our temporarily fixed $a \in A$. This implies that $\sup(B) = t \leq u - a$ for our temporarily fixed $a \in A$.

(c) Finally, show $\sup(A + B) = s + t$.

By part (b), $t = u - a$ for any upper bound u of $A + B$ and any temporarily fixed $a \in A$. Thus $a \leq u - t$ for any $a \in A$. Hence $s \leq u - t$, which can be rewritten as $s + t \leq u$. This shows that any upper bound u is at least as large as the upper bound $s + t$, so $s + t = \sup(A + B)$.

3. (Exercise 1.3.9.)

(a) If $\sup(A) < \sup(B)$, show that there exists an element $b \in B$ that is an upper bound for A.

Suppose that $\sup(A) = s < t = \sup(B)$. Choose $\varepsilon = t - s > 0$. By Lemma 1.3.8 there is an element $b \in B$ such that $b > t - \varepsilon = t - (t - s) = s$. Thus, $b \in B$ and $\sup(A) = s < b$, as desired.

(b) Give an example to show that this is not always the case if we only assume $\sup(A) \leq \sup(B)$.

If $A = B = (0, 1)$, then $\sup(A) = 1 = \sup(B)$, so $\sup(A) \leq \sup(B)$. But there is not $b \in B$ that is an upper bound for A, since $b \in B$ implies $b < 1$, but all upper bounds of A are $\geq \sup(A) = 1$.