Solutions to HW 1.

1. Show that the coimage of a function is a partition of the domain of the function.

Recall that the coimage of the function $f\colon A \to B$ is the set $\{f^{-1}(b) \mid b \in B\}$ of nonempty fibers of f.

Recall also that a partition of a set A is a set of nonempty subsets of A, say $P = \{A_i \mid i \in I\}$, such that

(1) $\bigcup_{i\in I} A_i = A$, and
(2) if $A_i \neq A_j$, then $A_i \cap A_j = \emptyset$.

The sets A_i are called the **cells** of the partition P.

Thus, we have to show that the fibers of f are the cells of a partition of A. This means that we have to show that

(1) $\bigcup_{b\in B} f^{-1}(b) = A$, and
(2) if $f^{-1}(b) \neq f^{-1}(c)$, then $f^{-1}(b) \cap f^{-1}(c) = \emptyset$.

For Item (1), each fiber is a subset of A, so $\bigcup_{b\in B} f^{-1}(b) \subseteq A$. To show the reverse inclusion, choose $a_0 \in A$ arbitrarily and let $b_0 = f(a_0)$. Then $a_0 \in f^{-1}(b_0) \subseteq \bigcup_{b\in B} f^{-1}(b)$. Since a_0 was chosen arbitrarily, this shows that $A \subseteq \bigcup_{b\in B} f^{-1}(b)$. Together, the two inclusions show that $\bigcup_{b\in B} f^{-1}(b) = A$.

We prove the contrapositive of Item (2). Assume that $f^{-1}(b)\cap f^{-1}(c) \neq \emptyset$, and hence that there is some $a \in f^{-1}(b) \cap f^{-1}(c)$. Since $a \in f^{-1}(b)$ we have $f(a) = b$. Since $a \in f^{-1}(c)$ we have $f(a) = c$. Thus $b = f(a) = c$, showing that $b = c$. It follows that the fiber $f^{-1}(b)$ over b is equal to the fiber $f^{-1}(c)$ over the equal element c. That is, $f^{-1}(b) = f^{-1}(c)$, completing the proof of the contrapositive.

2. (Exercise 1.2.13.) For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$S_n : \qquad (A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

We prove the statement S_n by induction on n.

(Base Case, n=1) This is the statement $A_1^c = A_1^c$, which is a tautology.
For the inductive step we also need the case $n = 2$, so we should prove that as a second base case. However, the problem statement says we can assume the truth of Exercise 1.2.5, which is S_2.

(Inductive Step) We assume that S_k is true and derive S_{k+1}.
Let $B = A_1 \cup A_2 \cup \cdots \cup A_k$ for this argument.

$$\begin{aligned}
(A_1 \cup \cdots \cup A_k \cup A_{k+1})^c &= (B \cup A_{k+1})^c && \text{(Defn. of } B\text{)} \\
&= B^c \cap A_{k+1}^c && (S_2) \\
&= (A_1^c \cap \cdots \cap A_k^c) \cap A_{k+1}^c && (S_k) \\
&= A_1^c \cap \cdots \cap A_k^c \cap A_{k+1}^c && \text{(Associativity of } \cap\text{)}
\end{aligned}$$

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i = \emptyset$ fails.

Choose half open real intervals $B_k = [k, \infty)$. Then $n \in \bigcap_{i=1}^{n} B_i$, so $\bigcap_{i=1}^{n} B_i \neq \emptyset$. On the other hand $\bigcap_{i=1}^{\infty} B_i = \emptyset$. (For a similar example, see Example 1.2.2 of the book.)

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

One can show that the sets $(\bigcup_{i=1}^{\infty} A_i)^c$ and $\bigcap_{i=1}^{\infty} A_i^c$ are equal by showing that they have the same elements.

$$\begin{aligned}
x \in (\textstyle\bigcup_{i=1}^{\infty} A_i)^c &\leftrightarrow x \notin (\textstyle\bigcup_{i=1}^{\infty} A_i) \\
&\leftrightarrow \forall i\, (x \notin A_i) \\
&\leftrightarrow \forall i\, (x \in A_i^c) \\
&\leftrightarrow x \in \textstyle\bigcap_{i=1}^{\infty} A_i^c.
\end{aligned}$$

3. (Exercise 1.3.2.) Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

$B = \{0\}$. ($\inf B = 0 = \sup B$.)

(b) A finite set that contains its infimum but not its supremum.

The request is impossible. Any set that has a supremum is nonempty, and any finite nonempty set must contain its infimum. Thus, either a finite set is empty, in which case it has no infimum or supremum, or it is not, in which case it contains both its infimum and supremum.
To see that a finite nonempty set must contain its infimum, we argue the contrapositive: if a nonempty set does not contain its infimum, then it is infinite. Let F be a nonempty set that does not contain its infimum. Choose any $a_0 \in F$; it cannot be the infimum. Therefore there is some $a_1 \in F$ such that $a_0 > a_1$. Now a_1 is not the infimum, so the same argument shows that there is some $a_2 \in F$ such that $a_0 > a_1 > a_2$. This process can be continued indefinitely to produce infinitely many elements of F: $a_0 > a_1 > a_2 > \cdots$.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$