Solutions to HW 10.

1. (Exercise 6.3.1.) Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.

Since the g_n's are increasing, for any $c \in [0,1]$ we have $|g_i(c)| \leq g_i(1) = 1/i$. Thus, given $\varepsilon > 0$, choose $N > 0$ so that $1/N < \varepsilon$. For $i > N$ we have $|g_i(c) - 0| \leq 1/i < 1/N < \varepsilon$. This shows that $(g_n(c)) \to 0$ for any c, so (g_n) converges pointwise to the zero function on $[0,1]$ $(g(x) \equiv 0)$.

In fact, since we were able to choose N in a way that did not depend on c, the preceding argument even shows that the convergence $(g_n) \to 0$ is uniform.

Since g is the zero function, it is differentiable with derivative 0 on $[0,1]$.

(b) Now, show that (g_n') converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g_n'$ and compare h and g'. Are they the same?

$g_n'(x) = x^{n-1}$. For $c \in [0,1)$ we have $\lim_{n\to\infty} g_n(c) = \lim_{n\to\infty} c^{n-1} = 0$, but for $c = 1$ we have $\lim_{n\to\infty} g_n(1) = \lim_{n\to\infty} 1 = 1$. The convergence cannot be uniform, since the limit of the continuous g_n's is not continuous. $h(1) = 1 \neq 0 = g'(1)$.

2. (Exercise 6.4.7(a)) Let $f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$ Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

Let $f_k(x) = \frac{\sin(kx)}{k^3}$, so that $f_k'(x) = \frac{\cos(kx)}{k^2}$. Then $|f_k'(x)| \leq \frac{1}{k^2} = M_k$, and $\sum \frac{1}{k^2}$ converges, so by the Weierstrass M-test $\sum f_k'(x)$ converges absolutely and uniformly to some function $g(x)$.

Now choose $c = 0$ and evaluate $\sum f_k(x)$ at c: we get $\sum f_k(0) = \sum 0 = 0$, so by the Term-by-Term DIfferentiability Theorem (Theorem 6.4.3) the series $\sum f_k(x)$ converges uniformly to $f(x)$, which is differentiable and satisfies $f'(x) = g(x)$. By the Uniform Limit Theorem, this derivative $g(x)$ is continuous, since it is the uniform limit of the continuous functions $\sum_{k=1}^{n} f_k'(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k^2}$.

3. (Exercise 7.2.1.) Let f be a bounded function on $[a,b]$, and let P be an arbitrary partition of $[a,b]$. First, explain why $U(f) \geq L(f,P)$. Now, prove Lemma 7.2.6.

(Done in class on November 30.)

By Lemma 7.2.4, $U(f,Q) \geq L(f,P)$ for any Q, so $U(f) = \inf_Q(U(f,Q)) \geq L(f,P)$, so $U(f) \geq L(f,P)$. This holds for any P, so $U(f) \geq \sup_P(L(f,P)) = L(f)$. This proves Lemma 7.2.6.