

## The types of $\langle \delta, \theta \rangle$ -minimal sets

Here  $\delta$  and  $\theta$  are compatible equivalence relations (congruences) of a finite algebra  $\mathbb{A}$ . In all results of this document, we let  $\mathbb{C}$  be the algebra  $(\mathbb{A}_A)|_U$  for some  $\langle \delta, \theta \rangle$ -minimal set.

### Lemma 3.1. (The Twin Lemma)

Let  $\mathbb{C}$  be minimal with respect to  $\langle \delta, \theta \rangle$ , and let  $B$  denote the body of  $\mathbb{C}$ . Suppose that  $f$  and  $g$  are unary body-twin polynomials of  $\mathbb{C}$  such that  $f$  is a permutation but  $g$  is not. Then  $B$  is a single  $\theta$ -class, which is a union of two  $\delta$ -classes, and  $\mathbb{C}$  has a binary polynomial that is a semilattice operation on  $B/\delta|_B$ .

For reasons that will become clear, we refer to the situation where  $\mathbb{C}$  has unary body-twin polynomials  $f$  and  $g$  where  $f$  is a permutation but  $g$  is not as “the nonabelian case”. The alternative case is the “abelian case”. The Twin Lemma says something about the structure of  $\mathbb{C}$  only in the nonabelian case.

The structural information of the Twin Lemma in the nonabelian case is refined in Lemmas 4.15 and 4.17 of Hobby-McKenzie. The structural information in the case where  $\langle \delta, \theta \rangle$  is abelian but not strongly abelian is provided by Lemmas 4.20 and 4.27 of Hobby-McKenzie. We don’t know much about the strongly abelian case, but some information is given by Theorem 3.4 of the “Easy Way” paper.

### Lemma 4.15. (slightly modified)

(Case where  $\mathbb{C}$  has body twins  $f$  and  $g$  where one is a permutation and the other is collapsing.) Let  $N$  be the unique trace of  $\mathbb{C}$ . There is an element  $1 \in N$  and a polynomial  $p(x, y) \in \text{Pol}_2(\mathbb{C})$  satisfying the following:

- (1)  $N = I \cup O$  (disjoint union), where  $I$  and  $O$  are  $\delta$ -equivalence classes and  $I = \{1\}$ .
- (2)  $p(N, N) \subseteq N$ , and  $\langle N, p \rangle / \delta|_N$  is polynomially equivalent to a 2-element semilattice with unit  $1/\delta|_N M$ .
- (3)  $p(1, x) = p(x, 1) = p(x, x) = x$  for  $x \in C$ .
- (4) For all  $x \neq 1$ ,  $u \in O$ ,  $p(u, x) \equiv_\delta p(x, u) \equiv_\delta p(x, x) = x$ .
- (5) For all  $x, y \in C$ ,  $p(x, p(x, y)) = p(x, y)$ .

If the structure induced on  $N$  is richer than a semilattice we can say more.

### Lemma 4.17. (modified)

Assume that the clone of  $\mathbb{C}|_N$  is richer than a 2-element semilattice. Then there exist  $p, q \in \text{Pol}_2(\mathbb{C})$  satisfying the following:

- (1)  $N = \{1, 0\}$  where 0 and 1 are  $\delta$ -inequivalent.
- (2)  $p(N, N) \subseteq N$ ,  $q(N, N) \subseteq N$ , and  $\langle N, p, q \rangle$  is polynomially equivalent to a 2-element lattice with unit 1 and zero 0.
- (3) For  $x \in C$ ,  $p(1, x) = p(x, 1) = p(x, x) = x = q(x, x) = q(x, 0) = q(0, x)$ .
- (4) For all  $x \notin N$ ,  $p(0, x) \equiv_\delta p(x, 0) \equiv_\delta p(x, x) = x = q(x, x) \equiv_\delta q(x, 1) \equiv_\delta q(1, x)$ .
- (5) For all  $x, y \in C$ ,  $p(x, p(x, y)) = p(x, y)$  and  $q(x, q(x, y)) = q(x, y)$ .

**Lemma 4.20. (modified)**

Assume that  $\langle \delta, \theta \rangle$  is abelian but not strongly abelian. Let  $B$  be the  $\langle \delta, \theta \rangle$ -body. Then there exist  $d \in \text{Pol}_3(\mathbb{C})$  satisfying the following:

- (1)  $d(x, x, x) = x$  on  $C$ .
- (2)  $d(x, x, y) = y = d(y, x, x)$  for  $x \in B$  and  $y \in C$ .
- (3) For every  $a, b \in B$ , the polynomials  $d(a, b, x), d(a, x, b), d(x, a, b)$  are permutations of  $C$ .
- (4)  $B$  is closed under  $d$ .
- (5) Any two traces are isomorphic.

Define the twin congruence on  $\mathbb{C}$  by  $(a, b) \in \beta$  iff  $f(x, a)$  is a permutation of  $C$  iff  $f(x, b)$  is a permutation of  $C$ . In the nonabelian case  $\beta \wedge \theta = \delta$ . In the abelian but not strongly abelian case we have:

**Lemma 4.27. (modified)**

Assume that  $\langle \delta, \theta \rangle$  is abelian but not strongly abelian. Let  $B$  be the  $\langle \delta, \theta \rangle$ -body and  $T$  be the  $\langle \delta, \theta \rangle$ -tail.

- (1)  $B$  is a  $\beta$ -class.
- (2) If  $f \in \text{Pol}_n(\mathbb{C})$ ,  $f(B^n) \subseteq B$  or  $f(B^n) \subseteq T$ .
- (3) The largest congruence  $\theta$  included in  $B^2 \cup T^2$  satisfies
  - (a) For all congruences  $\lambda$ , either  $\theta \leq \lambda$  or  $\theta \vee \delta \geq \lambda$ .
  - (b) Any  $\langle \gamma, \lambda \rangle$  with  $\lambda \leq \theta$  and  $\gamma \not\leq \theta$  is nonabelian.

**Lemma.** Assume that  $\langle \delta, \theta \rangle$  is strongly abelian. Any two traces are isomorphic. If  $N$  is a trace, then  $\mathbb{C}|_N / \delta|_N$  is polynomially equivalent to a  $G$ -set.