Some theorems about series.

Theorem 1. *(**Uniform Limit Theorem**, Thm 6.2.6 of the book)*
("The uniform limit of continuous functions is continuous".)
If (f_n) is a sequence of functions that are continuous on $[a,b]$, and $(f_n) \to f$ uniformly, then f is continuous on $[a,b]$.

Theorem 2. *(Thm 6.3.3 of the book)*
Let (f_n) be a sequence of functions differentiable on $[a,b]$, and assume $(f_n') \to g$ uniformly on $[a,b]$. If there is a point $c \in [a,b]$ such that $(f_n(c))$ converges, then (f_n) converges uniformly to some limit function f, which is differentiable and satisfies $f' = g$.

We can rewrite these theorems for sequences so that they make sense for series.

Theorem 3. *(Thm 6.4.2 of the book) If (f_n) is a sequence of functions that are continuous on $[a,b]$, and $\sum f_n$ converges to f uniformly, then f is continuous on $[a,b]$.*

Theorem 4. *(Thm 6.4.3 of the book)*
Let (f_n) be a sequence of functions differentiable on $[a,b]$, and assume $\sum f_n'$ uniformly on $[a,b]$. If there is a point $c \in [a,b]$ such that $\sum f_n(c)$ converges, then $\sum f_n$ converges uniformly on $[a,b]$ and $(\sum f_n)' = \sum f_n'$.

A flexible tool for checking uniform convergence of series is the **Weierstrass M-test**:

Theorem 5. *(Cor 6.4.5 of the book) Suppose that $|f_n(x)| \leq M_n$ on a set $A \subseteq \mathbb{R}$ and that $\sum M_n$ converges. Then $\sum f_n(x)$ converges absolutely and uniformly on A.*

Practice Problems.

(1) Using the Weierstrass M-test, explain why if $\sum a_n x^n$ converges at some point $R > 0$, then it converges absolutely and uniformly on $[-c, c]$ for any $c < R$.

(2) Using the previous problem, show that $\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)' = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$.

(3) Let $f_n(x)$ be the continuous, piecewise-linear function defined on $[0,1]$ by

$$f_n(x) = \begin{cases} -x, & x \in [0, \frac{1}{n+1}] \\ nx - 1, & x \in [\frac{1}{n+1}, \frac{1}{n}] \\ 0, & x \in [\frac{1}{n}, 1]. \end{cases}$$

The series $\sum_{n=0}^{\infty} f_n(x)$ converges pointwise to a discontinuous function. What part of the Weierstrass M-test fails to hold?

(4) Apply the Weierstrass M-test and the Uniform Limit Theorem to show that the blancmange function is continuous.