Limits need not commute with each other.

It is easy to fill an infinite 2×2 array with numbers so that the limit along each row is 0 while the limit along each column is 1. For example, you could fill cell $A[m, n]$ with 1 if $m \geq n$ and with 0 if $m < n$, as follows:

$m \backslash n$	1	2	3	4	$\cdots$
1	1	0	0	0	$\cdots$
2	1	1	0	0	
3	1	1	1	0	
4	1	1	1	1	
$\vdots$					

Then

$$\lim_{m\to\infty} \lim_{n\to\infty} A[m, n] = \lim_{m\to\infty} 0 = 0 \neq 1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty} \lim_{m\to\infty} A[m, n].$$

This shows that, in general, limiting processes typically do not commute with each other. Therefore it should not be a surprise to learn that the limiting processes of differentiation and integration need not commute with each other or with themselves. Luckily they do commute in sufficiently nice situations.

Clairaut's Theorem.

Theorem 1. *(Clairaut's Theorem.) If $f(x, y)$ is defined in an open ball B and f_{xy} and f_{yx} exist and are continuous on B, then*

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

on B.

Related Example. Let

$$f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then $\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)(0, 0) = 1$, while $\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)(0, 0) = -1$.

Leibniz Integral Rule.

Theorem 2. *(Leibniz Integral Rule, Special Case.) If $f(x, y)$ and $f_x(x, y)$ exists and are continuous on a rectangle $[x_0, x_1] \times [a, b]$, then*

$$\frac{d}{dx}\int_a^b f(x, y)\, dy = \int_a^b \frac{\partial}{\partial x} f(x, y)\, dy$$

on $[x_0, x_1]$.

Related Example. Let

$$f(x,y) = \begin{cases} \frac{xy^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

When evaluated at $x = 0$, the left side of

$$\frac{d}{dx}\int_a^b f(x,y)\, dy \stackrel{?}{=} \int_a^b \frac{\partial}{\partial x} f(x,y)\, dy$$

can be shown to be 1/2 while the right side can be shown to be 0.

Fubini's Theorem.

Theorem 3. *(Fubini's Theorem.) If $f(x,y)$ is continuous on $[a,b] \times [c,d]$, then*

$$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy.$$

Related Example. Let

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then $\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx = \pi/4$, while $\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = -\pi/4$.