Review of some elements of Math 2001, Discrete Math.

Everyone knows that

- You can't divide by zero.
- $0 \neq 1$.
- $1 + 1 = 2$.
- for any natural numbers x and y, $x + y = y + x$.
- $1 = .999\ldots$ in $\mathbb{R}$.

But what are the correct explanations for these facts?

For example, why does $x + y$ equal $y + x$ for any two natural numbers x and y? When you were a child, a teacher might have explained that if you put x apples in a bag and then add y more apples, then you will have $x + y$ apples in the bag. Now if you take out the last y apples, then the remaining x apples, you will have removed $y + x$ apples. But you removed the same number you put in, so $x + y = y + x$. Is this a correct explanation?

No it isn't. The correct explanation must refer to the definition of $+$. You should have learned in Math 2001 that $+$ is defined on $\mathbb{N}$ by recursion. The proof that $\forall x \forall y (x + y = y + x)$ holds must therefore be a proof by induction.

Similarly, when a child, you might have learned that the reason $1 + 1 = 2$ is that if you put one apple in a bag and then put another apple in the bag, you will find after counting that you have two apples in the bag. This explanation is also wrong. The correct explanation of why $1 + 1 = 2$ must refer to the definitions of 1, $+$, $=$ and 2.

The prerequisite course Math 2001 (Discrete Math) is designed to be a "transition to higher math" course, presented in the context of Discrete Mathematics. The "transition to higher math" part of Math 2001 involved learning

- that mathematics is created to be *well founded*,
- that the most commonly used foundation for mathematics is set theory, and
- how to read, write, and interpret formal sentences.[1]

I. Well foundedness. One goal of mathematicians is to organize mathematics so that it is *well founded*. Each time a new concept is introduced, it is defined in terms of more primitive concepts. These more primitive concepts are themselves defined in terms of still more primitive concepts. The process of unravelling meaning continues *until it stops*. And it does stop, when one finally reaches the undefined terms of mathematics, which are

- set, and
- set membership.

Similarly, the claims of mathematics are proved using more primitive claims, which are themselves proved in terms of more primitive claims. The procedure of unravelling provability stops when when one finally reaches the unprovable claims accepted in mathematics, which are the axioms of set theory.

The idea that mathematical definitions and claims can be ordered by the relation of primitivity, and that the unravelling procedure *does terminate*, is what we mean when we say that mathematics is intended to be well founded.

[1]You will have also learned some traditional Discrete Mathematics topics, like formulas for counting and facts about graphs, but that is not the part of the course used as a prerequisite for other courses.

Exercise.
What is the definition of *finite set*?

First-level answer.
x is a *finite set* if there is a <u>*natural number*</u> $k \in \mathbb{N}$ and a <u>*bijection*</u> $f : k \to x$.

A person who already knows what a natural number is and what a bijection is will understand through this definition what a finite set is. Both *natural number* and *bijection* are etymologically more primitive than *finite set*.

More fully unravelled answer.

(1) (finite set) x is a finite set if there is a <u>*natural number*</u> $k \in \mathbb{N}$ and a <u>*bijection*</u> $f : k \to x$.
  (a) (natural number) A natural number is a member of the <u>*set of natural numbers*</u>.
    (i) (set of natural numbers) The set of natural numbers is the <u>*least inductive set*</u>.
      (A) (inductive set) A set x is inductive if $\underline{0} \in x$ and, for all $y \in x$ we have $\underline{y \cup \{y\}} \in x$.
      (B) (least (inductive set)) The least inductive set is the <u>*intersection*</u> of all inductive sets.
  (b) (bijection) A bijection from a set x to a set y is a <u>*function from x to y*</u> that is <u>*invertible*</u>.
    (i) (function from x to y) A function from x to y is a <u>*relation from x to y*</u> which satisfies the <u>*function rule*</u>.
      (A) (relation from x to y) A relation from x to y is a <u>*subset*</u> of the <u>*Cartesian product $x \times y$*</u>.
      (B) (function rule) A relation R from x to y satisfies the function rule if for every $u \in x$ there exists exactly one $v \in y$ such that $\underline{(u, v)} \in R$.
    (ii) (invertible function) A function $f : x \to y$ is invertible if there is a function $g : y \to x$ such that the <u>*compositions*</u> $g \circ f : x \to x$ and $f \circ g : y \to y$ are both <u>*identity functions*</u>.
      (A) (composition of functions) The composition of $f : x \to y$ and $g : y \to z$ is

$$g \circ f := \underline{\{(u, v) \in x \times z \mid \exists w((w \in y) \text{ and } ((u, w) \in f) \text{ and } ((w, v) \in g))\}}.$$

      (B) (identity function) The identity function on x is

$$\mathrm{id}_x := \underline{\{(u, u) \in x \times x \mid u \in x\}}.$$

This is still not the completely unravelled answer, but Latex has a limit on the depth of enumerated lists. If you wanted to continue unravelling, you would need to provide definitions of: 0, union, intersection, subset, ordered pair, Cartesian product, power set, and you would have to explain the meaning of sets described using set-builder notation. You would also have to provide some justifications, for example: Why is there an inductive set? Why is the intersection of inductive sets inductive? Why is the Cartesian product of two sets a set? Why are $g \circ f$ and id_x functions? ETC.

This Exercise shows that fully comprehending the basic notion of a "finite set" requires first comprehending layers and layers of more primitive notions. In particular, "0" is more primitive than "inductive set"; "inductive set" is more primitive than "$\mathbb{N}$"; "$\mathbb{N}$" is more primitive than "natural number"; "natural number" is more primitive than "finite set".

[Side note: there is an oddity here that you might have noticed. If $\mathbb{N}$ denotes the set of natural numbers and $k \in \mathbb{N}$ is an individual natural number, then the single number k is etymologically more complex than the set $\mathbb{N}$. This is because $\mathbb{N}$ is defined by a property and not defined by reference to its elements.[2]]

II. Reading, writing, and interpreting formal sentences. In your Discrete Math course, you must have learned how to read, write, and interpret formal sentences, like $\forall x \exists y(x = y)$. "Learning how to read, write, and interpret formal sentences" means learning:

(1) Which symbols may be used. (E.g. $x, y, \forall, \exists, \vee, \wedge, \rightarrow, \ldots$, ETC.)
(2) Correct sentence formation rules. (E.g., why is $\forall x \exists y(x = y)$ correctly formed, while $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x = y)$ incorrectly formed?)
(3) How to determine the meaning of a sentence. (E.g., is $\forall x \exists y(x = y)$ true in $\mathbb{R}$? Is it true in $\mathbb{N}$?)
(4) Logical equivalences. (E.g., $\neg \forall x A \equiv \exists x \neg A$, or $\exists x(A \text{ or } B) \equiv ((\exists x A) \text{ or } (\exists x B))$.)

Here are some refresher exercises.

Exercises.

(1) Is $\forall x \exists y(x = y)$ true in $\mathbb{R}$? Explain.

(2) Is $\forall a\ \exists b\ \forall c\ \exists d\ (a^2 + b^2 = c^2 + d^2)$ true in $\mathbb{R}$? Is it true in $\mathbb{C}$? Explain.

(3) Suppose $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto 1$. Is the following statement true in $\mathbb{R}$? Explain.

$$\exists x_0\ \forall \varepsilon\ \exists \delta\ \forall x\ ((\varepsilon > 0) \rightarrow ((0 < |x - x_0| < \delta) \rightarrow (|f(x) - f(x_0)| < \varepsilon)))$$

(4) Write a sentence that is true in a set if and only if the set contains exactly 2 members.

(5) Write a sentence that is meaningful for both $\mathbb{N}$ and $\mathbb{R}$, and is true in $\mathbb{N}$ but false in $\mathbb{R}$.

[2] It is an etymological mistake to define $\mathbb{N}$ to be $\{0, 1, 2, \ldots\}$ (which is ambiguous), and an even bigger mistake to define $\mathbb{N}$ to be $\{1, 2, 3, \ldots\}$ (which is not an inductive set under the modern meaning of that phrase).