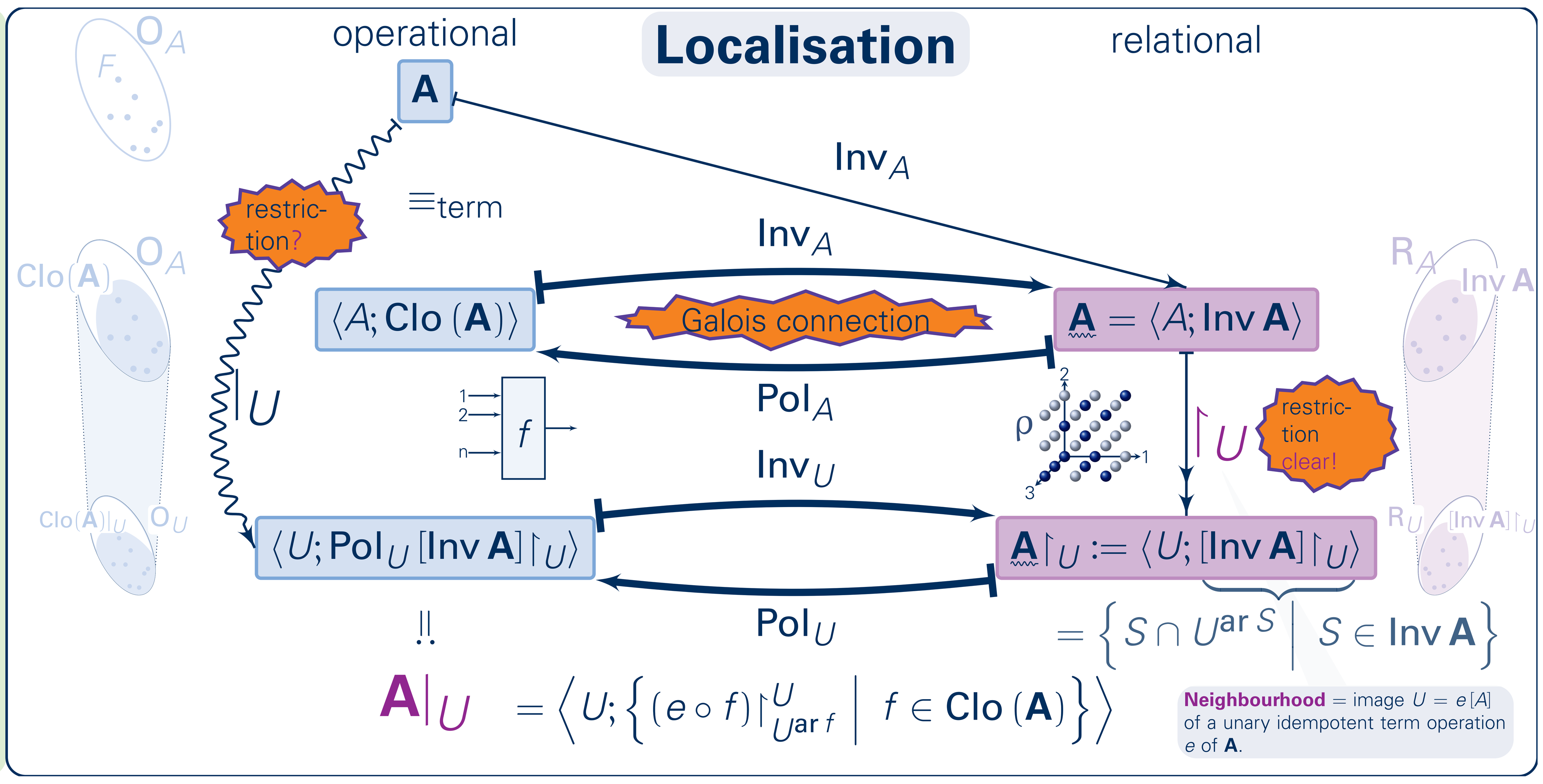
Decomposition of Algebras via Relations

RST – a Relational Structure Theory for finite algebras

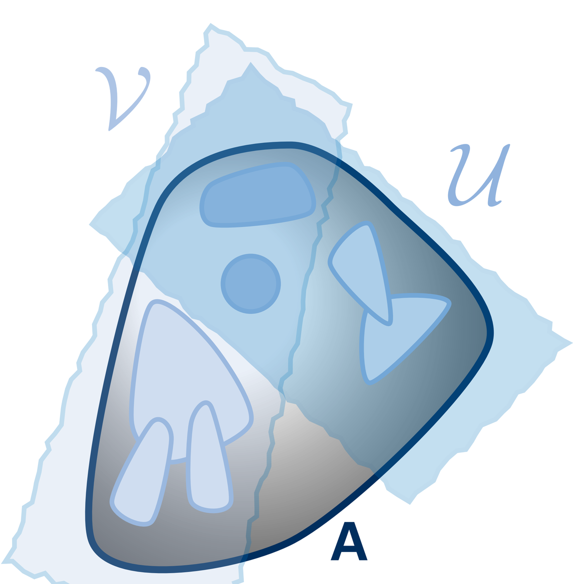
Goal

RST aims at decomposing finite algebras $\mathbf{A} = \langle A; F \rangle$, i.e. finite sets A carrying a structure consisting of a set F of finitary operations. This structure is restricted to certain subsets $U \subseteq A$, called *neighbourhoods*, ideally yielding smaller algebras.
The objective is to develop a localisation theory for algebraic structures by considering their clone $\mathrm{Inv}\,\mathbf{A}$ of invariant relations, restricting it, and finally enabling reconstruction of the original structure up to term equivalence.

Localisation



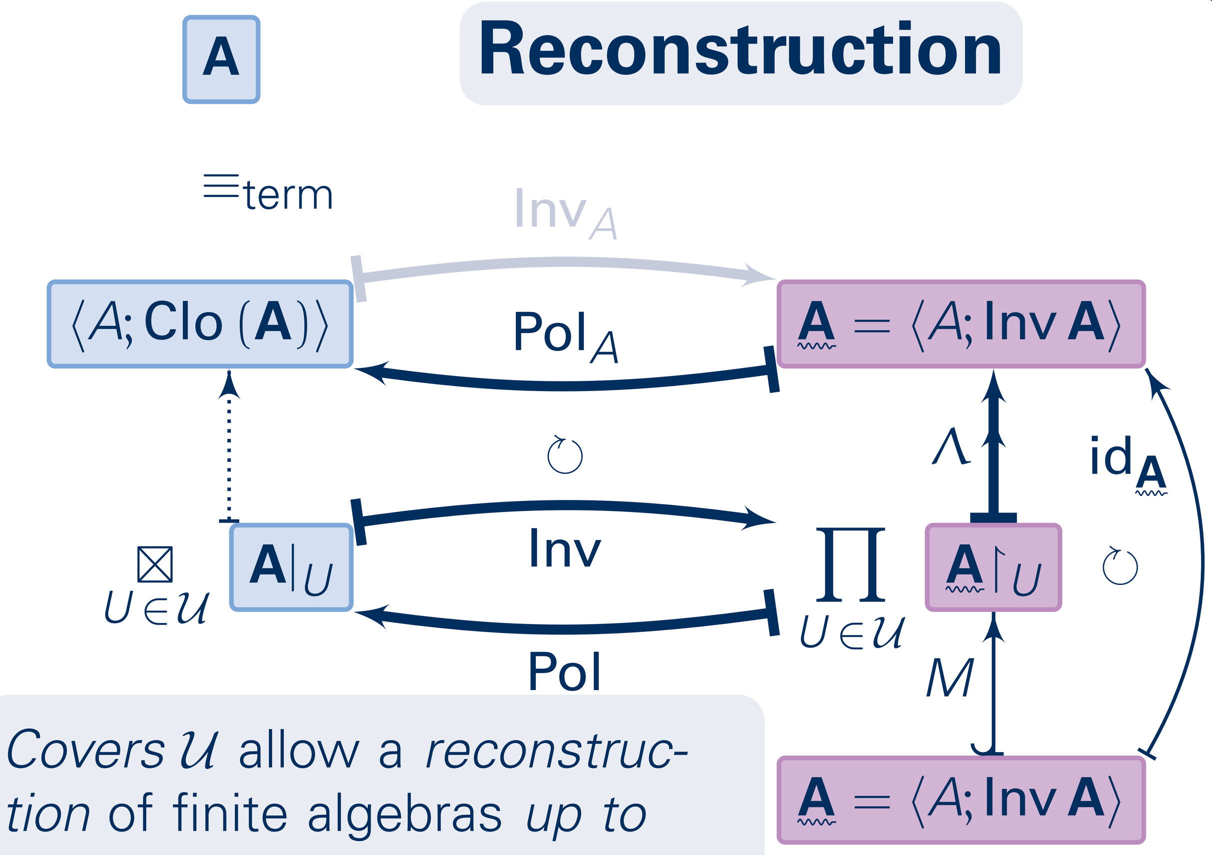
Covers



Covers = collections $\mathcal{U}$ of neighbourhoods to *separate* invariant relations as follows: for all $m \in \mathbb{N}$ and $S, T \in \mathrm{Inv}^{(m)}\mathbf{A}$:

$$S \neq T \implies \exists U \in \mathcal{U}: \; S\restriction_U = T\restriction_U$$

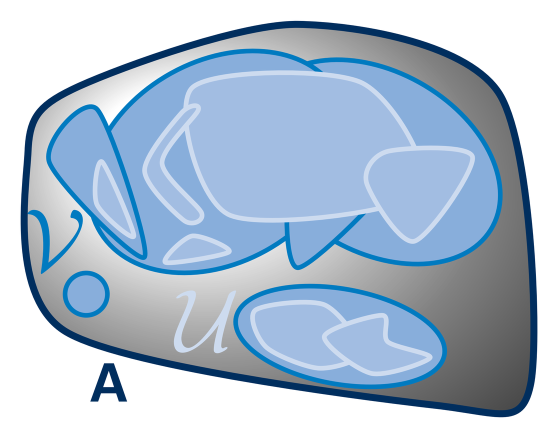
Reconstruction



Covers $\mathcal{U}$ allow a *reconstruction* of finite algebras *up to term equivalence* via a retraction Λ on the relational side, yielding categorical equivalences.

Refinement quasi-order

Which cover is the right one?

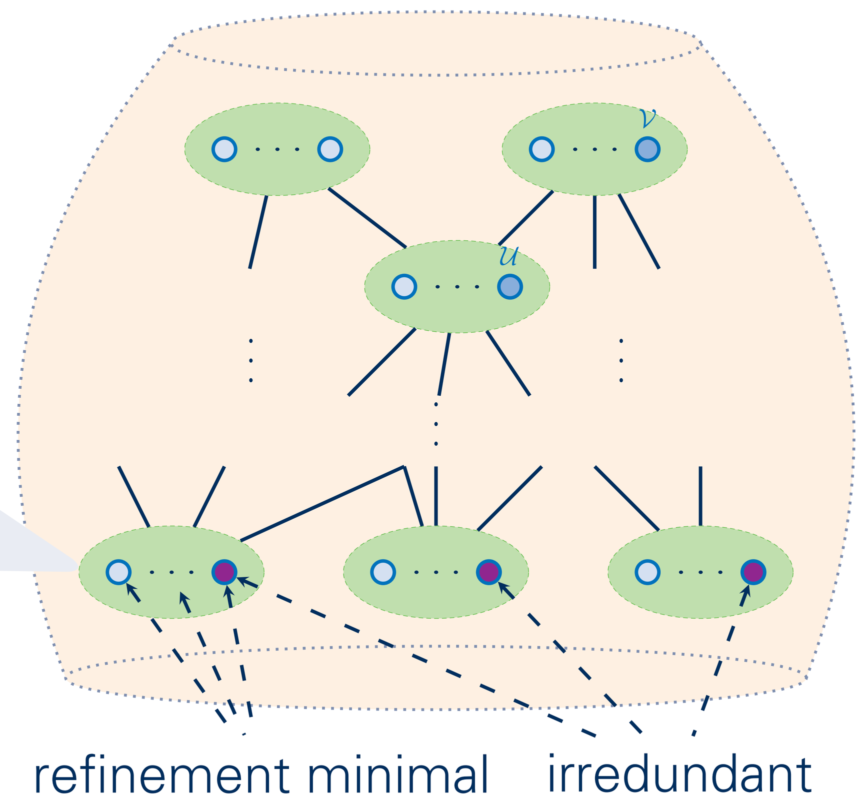


$\mathcal{U} \leq_{\mathrm{ref}} \mathcal{V}$ if each neighbourhood of $\mathcal{U}$ is contained in some neighbourhood of $\mathcal{V}$.

In general, there is an abundance of covers available to decompose an algebra. Using the refinement quasi-order, one can find optimal covers with particularly small neighbourhoods.

Quasi-ordered set of all covers

Nonrefinable covers are $\leq_{\mathrm{ref}}$*-minimal irredundant covers*. Every block of $\leq_{\mathrm{ref}}$-equivalent $\leq_{\mathrm{ref}}$-minimal covers contains precisely one cover without unnecessary neighbourhoods, a so-called *irredundant* cover.
For every finite algebra there *exists* a nonrefinable cover, and it is *unique up to isomorphism*. Its neighbourhoods are *irreducible*, a property characterised by the the set of unary term operations forming a subpower.



Summary and prospects

By restriction to sufficiently many irreducible neighbourhoods, every finite algebra can be decomposed in such a way, that it is, in principle, i.e. up to term equivalence, completely reconstructible from the arising *irreducible algebras*. In this sense the latter constitute the building blocks of all finite algebras.
This theory, RST, based on ideas by Kearnes and Szendrei, is further developed within the research project "Zerlegung und Komposition algebraischer Strukturen" funded by the DFG (German Research Foundation). Results and areas of current research include an effective characterisation of nonrefinable covers, and relating concepts for single algebras to locally finite varieties.

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