The Group Determinant

The multiplication table of a group of size n is an $n \times n$ array, $[a_{g,h}]$ where $a_{g,h} = gh$. What is the determinant of this matrix? This doesn't make sense, since to evaluate a determinant you need to add and multiply values. But in 1880 Richard Dedekind asked a version of the question which does make sense. Introduce a set of commuting variables $T = \{t_g \mid g \in G\}$, replace all instances of g with t_g in the operation table of the group, and consider the table $[t_{gh}]$ to be a matrix over the function field $\mathbb{C}(T)$. Now the determinant makes sense, and it is a homogeneous polynomial of degree $|G|$ over $\mathbb{C}$. Question: what does this polynomial tell you about G?

Example 1. Suppose that $G = \{1, g\}$ is a 2-element group. After replacing the elements of the group table with commuting variables, we obtain $\begin{bmatrix} t_1 & t_g \\ t_g & t_1 \end{bmatrix}$, whose determinant is $\Theta(G) = t_1^2 - t_g^2 = (t_1 + t_g)(t_1 - t_g)$.

Instead of putting the variable t_{gh} in the (g, h)-position of the table, it is better to put $t_{gh^{-1}}$. This alters the situation only by interchanging the g and g^{-1} columns of the group table, so it effects the determinant only by ± 1. One reason for making the change is that it normalizes the determinant (the coefficient of the leading term $t_1^{|G|}$ is $+1$).

Example 2. The change described in the previous paragraph does not alter Example 1, so consider a 3-element group $G = \{1, g, g^2\}$. Then

$$\Theta(G) = \begin{bmatrix} t_1 & t_{g^2} & t_g \\ t_g & t_1 & t_{g^2} \\ t_{g^2} & t_g & t_1 \end{bmatrix} = t_1^3 + t_g^3 + t_{g^2}^3 - 3t_1 t_g t_{g^2}.$$

This polynomial factors over $\mathbb{C}$ as $(t_1 + t_g + t_{g^2})(t_1 + \omega t_g + \omega^2 t_{g^2})(t_1 + \omega^2 t_g + \omega t_{g^2})$ where ω is a primitive cube root of unity.

The factorization of the previous example had meaning for Dedekind. A *character* of a finite abelian group G is a homomorphism $\chi \colon G \to \mathbb{C}^*$ from G to the nonzero complex numbers under multiplication. Under pointwise multiplication the collection $\widehat{G}$ of all characters of G is a group isomorphic to G. The three characters of $G = \{1, g, g^2\}$ are the functions χ_1, χ_2, χ_3 determined on the generating set $\{g\}$ of G by $\chi_{i+1}(g) = \omega^i$. Using these characters, Dedekind could write $\Theta(G) = \prod_{i=1}^{3}(\chi_i(1)t_1 + \chi_i(g)t_g + \chi_i(g^2)t_{g^2})$ for the factorization of the group determinant in Example 2. This led to a conjecture, and then a theorem:

Theorem 3. *Let G be a finite abelian group with dual group $\widehat{G}$. The factorization of the group determinant is $\Theta(G) = \prod_{\chi \in \widehat{G}} P_\chi$ where $P_\chi = \left(\sum_{g \in G} \chi(g) t_g\right)$. (So, $\Theta(G)$ is a homogeneous polynomial of degree $|G|$, and it factors into $|G|$ homogeneous linear terms.)*

This answers the question of what the group determinant tells you about a finite abelian group: it tells you the characters of G.

Example 4. If $D_3 = \{1, r, r^2, f, rf, r^2f\}$ is the dihedral group, then $\Theta(G)$ is the product of the homogeneous factors $(t_1 + t_r + t_{r^2} + t_f + t_{rf} + t_{r^2f})$, $(t_1 + t_r + t_{r^2} - t_f - t_{rf} - t_{r^2f})$, and $(t_1^2 + t_r^2 + t_{r^2}^2 - t_1t_r - t_1t_{r^2} - t_rt_{r^2} - t_f^2 - t_{rf}^2 - t_{r^2f}^2 + t_ft_{rf} + t_ft_{r^2f} + t_{rf}t_{r^2f})^2$

The linear factors in this product are derived from homomorphisms $\chi \colon D_3 \to \mathbb{C}^*$ just as before, but what does the last squared factor of degree 2 mean? Dedekind worked on this question on and off from 1880 to 1896, and finally asked Frobenius (an expert in the theory of determinants) to consider it.

Frobenius solved the problem by generalizing Dedekind's definition of a character, and explaining how to compute these generalized characters.

Definition 5. A *character* of a finite group G is a function $\chi \colon G \to \mathbb{C}$ that can be factored as $G \xrightarrow{\rho} \mathrm{GL}(d, \mathbb{C}) = M_d(\mathbb{C})^* \xrightarrow{\mathtt{tr}} \mathbb{C}$ where ρ is a group homomorphism and $\mathtt{tr}$ is the trace map. Here d is called the *degree* of χ. A character χ is *irreducible* if the $\mathbb{C}$-subspace of $M_d(\mathbb{C})$ generated by $\rho(G)$ is all of $M_d(\mathbb{C})$. The *k-character* associated to a character χ is the function $\chi^{(k)} \colon G^k \to \mathbb{C}$ defined inductively by:

(i) $\chi^{(1)}(g) := \chi(g)$, and
(ii) $\chi^{(r)}(g_1, g_2, \ldots, g_r) = \chi(g_1)\chi^{(r-1)}(g_2, g_3, \ldots, g_r) - \chi^{(r-1)}(g_1 \cdot g_2, g_3, \ldots, g_r) - \chi^{(r-1)}(g_2, g_1 \cdot g_3, \ldots, g_r) - \cdots - \chi^{(r-1)}(g_2, g_3, \ldots, g_1 \cdot g_r)$.

Theorem 6. *Let G be a finite group, and let $\mathcal{X} = \{\chi_1, \ldots, \chi_m\}$ be a complete set of irreducible characters of G. Then $|\mathcal{X}|$ equals the class number of G, and the complete factorization of the group determinant is $\Theta(G) = \prod_{\chi \in \mathcal{X}} P_\chi$ where $P_\chi = \frac{1}{d!}\left(\sum_{\overline{g} \in G^d} \chi^{(d)}(\overline{g}) t_{\overline{g}}\right)^d$ if the degree of χ is d. Here if $\overline{g} = (g_{i_1}, g_{i_2}, \ldots, g_{i_d})$, then $t_{\overline{g}} = t_{g_{i_1}} t_{g_{i_2}} \cdots t_{g_{i_d}}$.*

Thus, Frobenius showed that $\Theta(G)$ determines the k-characters of G for all k. It has since been shown (by Formanek and Sibley (1991)) that $\Theta(G)$ detrmines G up to isomorphism. This was improved (by Hoehnke and Johnson (1992)) to show that the 1-, 2-, and 3-characters of G determine G up to isomorphism.