Solving the Quartic Equation.

Our goal is to learn how to find the roots of $ax^4+bx^3+cx^2+dx+e=0$, where $a \neq 0$. The first person to discover how to do this was Lodovico de Ferrari (who lived 1522-1565), but we will follow Descartes' Method. (Descartes lived 1596-1650, and published his method in 1637).

Stage 1. Divide by leading coefficient and depress the equation. The result will have the form

$$y^4 + py^2 + qy + r = 0.$$

Stage 2. Factor into quadratics. The factorization will look like

$$y^4 + py^2 + qy + r = (y^2 - sy + u)(y^2 + sy + v).$$

Since $(y^2 - sy + u)(y^2 + sy + v) = y^4 + (u + v - s^2)y^2 + (s(u - v))y + uv$, we can equate coefficients in the previous displayed line to obtain a system of polynomial equations

$$\begin{aligned} u + v - s^2 &= p \\ s(u - v) &= q \\ uv &= r \end{aligned}$$

In this system, s, u, v are unknowns while p, q, r are known values from Stage 1. If we solve this system for s, u, v we will have obtained a factorization of the depressed quartic into two quadratic factors.

Let's reduce the system of equations to a single equation in the variable s.

$$\begin{aligned} u + v &= p + s^2 \\ u - v &= q/s, \end{aligned}$$

so $u = \frac{1}{2}(p + s^2 + q/s)$ and $v = \frac{1}{2}(p + s^2 - q/s)$, hence

$$r = uv = \frac{1}{2}\left(p + s^2 + q/s\right) \cdot \frac{1}{2}\left(p + s^2 - q/s\right) = \frac{1}{4}\left((p + s^2)^2 - (q/s)^2\right),$$

or

$$4r = (p^2 + 2ps^2 + s^4) - q^2/s^2,$$

or

$$s^6 + 2ps^4 + (p^2 - 4r)s^2 - q^2 = 0.$$

Now let $S = s^2$. The above equation can be written as

$$S^3 + 2pS^2 + (p^2 - 4r)S - q^2 = 0.$$

This cubic is the *resolvent cubic.*

Using the Cardano Formula, solve $S^3 + 2pS^2 + (p^2 - 4r)S - q^2 = 0$. If the roots are $S = R_1, R_2, R_3$, then $s = \pm\sqrt{R_1}, \pm\sqrt{R_2}, \pm\sqrt{R_3}$. For a given value of s, say $s = \sqrt{R_1}$, we can solve for u and v:

$$\begin{aligned} u &= \tfrac{1}{2}(p + s^2 + q/s) \\ v &= \tfrac{1}{2}(p + s^2 - q/s). \end{aligned}$$

Once we have u, v, s, we have achieved our factorization:

$$y^4 + py^2 + qy + r = (y^2 - sy + u)(y^2 + sy + v).$$

Stage 3. Apply the quadratic formula to each of the quadratics $y^2 - sy + u = 0$, $y^2 + sy + v = 0$ to obtain the roots of the depressed quartic equation $y^4 + py^2 + qy + r = (y^2 - sy + u)(y^2 + sy + v) = 0$. The roots will be

$$\begin{aligned} y_1 &= \tfrac{s+\sqrt{s^2-4u}}{2}, \\ y_2 &= \tfrac{s-\sqrt{s^2-4u}}{2}, \\ y_3 &= \tfrac{-s+\sqrt{s^2-4v}}{2}, \\ y_4 &= \tfrac{-s-\sqrt{s^2-4v}}{2}. \end{aligned}$$

These can be used to solve the original (undepressed) equation.

Point to ponder. We found six possible values for s, but we used only one of them. What if we had used a different value? Could there be still more roots of the quartic?