Intersection multiplicity for plane curves.

Suppose that $F(x,y) = 0$ and $G(x,y) = 0$ are plane curves. Write $I_P(F,G)$ for the multiplicity of the intersection of the curves at a point P. The number $I_P(F,G)$ has the following properties.

(1) $I_P(F,G) = 0$ unless $F(P) = G(P) = 0$.
(2) If M is a linear change of variables, then $I_P(FM, GM) = I_{MP}(F,G)$.
(3) $I_{(0,0)}(x,y) = 1$.
(4) $I_P(F,G) = \infty$ if $F = F'H$ and $G = G'H$ for some polynomial factor H such that $H(P) = 0$.
(5) $I_P(F,G) = I_P(G,F)$
(6) $I_P(F, G_1G_2) = I_P(F,G_1) + I_P(F,G_2)$.
(7) $I_P(F + GH, G) = I_P(F,G)$.

Examples.

(a) By item (3), the intersection of the curve $y = 0$ (the x-axis) with $x = 0$ (the y-axis) has multiplicity 1 at $P = (0,0)$. That is, $I_P(x,y) = I_P(y,x) = 1$.

(b) Suppose that $F(x,y) = y - x^n$ and $G(x,y) = y$. Then for $P = (0,0)$ we have

$$I_P(y - x^n, y) = I_P(-x^n, y) = I_P(y, -x^n) = I_P(y, -1) + nI_P(y,x) = 0 + n \cdot 1 = n.$$

(c) Suppose that $F(x,y) = y - x^2$ and $G(x,y) = y - x^3$. Then for $P = (0,0)$ we have

$$\begin{aligned} I_P(y - x^2, y - x^3) &= I_P(x^3 - x^2, y - x^3) \\ &= I_P(y - x^3, x^3 - x^2) \\ &= I_P(y - x^3, x - 1) + 2I_P(y - x^3, x) \\ &= 0 + 2I_P(y,x) = 2. \end{aligned}$$

To find the intersection multiplicity of curves in the projective plane, apply a projective transformation if necessary to move the intersection point to a finite point, say $(0,0)$, and calculate the multiplicity there according the above rules.

Exercise. Find the points of intersection and the multiplicities of intersection of the curve $y = x^n$ with each of the lines $x = 0$, $y = 0$ and $z = 0$. (This is three problems.)