HW 9: solution sketches

(1) Suppose that $f(x)$ is a polynomial of degree greater than 1. Explain how to find the points at infinity on the curve $y = f(x)$.

Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $n > 1$. Then the homogenization of $y = f(x)$ is $yz^{n-1} = a_n x^n + a_{n-1} x^{n-1} z + \cdots + a_1 x z^{n-1} + a_0 z^n$. The points at infinity are found by setting $z = 0$, which yields $0 = a_n x^n$. From this it follows that $x = 0$. Thus a point at infinity has homogeneous coordinates

$$\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This shows that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the only point at infinity on $y = f(x)$.

(2) Is there a (complex) polynomial $F(x, y)$ such that the projective completion of the curve defined by $F(x, y) = 0$ has no points at infinity?

No. Bézout's Theorem implies that any line must intersect any curve in the projective plane.

(3) Everybody knows that 2 points are sufficient to determine a line. How many points are sufficient to determine an irreducible conic in the projective plane over $\mathbb{C}$?

Five points determine a conic.

Four points are insufficient, in general, since it is easy to write down two conics that intersect in four distinct points — those four points do not determine which of the two conics you are interested in.

But five points are enough, by Bézout's Theorem. Given 5 points, it is impossible for two different irreducible conics to pass through them, since Bézout's Theorem only allows four points to lie on two different irreducible conics.