HW 7: solution sketches

(1) Find the roots of $z^4 + 3z^2 + 6z - 5 = 0$.

To find the roots of this quartic, we first try to factor it into quadratics:

$$z^4 + 3z^2 + 6z - 5 = (z^2 - az + b)(z^2 + az + c).$$

Multiplying the right hand side and equating coefficients we get

$$\begin{aligned} b + c - a^2 &= 3 \\ a(b - c) &= 6 \\ bc &= -5. \end{aligned}$$

The first two of these equations may be rewritten as

$$\begin{aligned} b + c &= 3 + a^2 \\ b - c &= 6/a, \end{aligned}$$

or

$$\begin{aligned} b &= \tfrac{1}{2}\left((a^2+3) + \tfrac{6}{a}\right) \\ c &= \tfrac{1}{2}\left((a^2+3) - \tfrac{6}{a}\right). \end{aligned}$$

Thus

$$\begin{aligned} -5 = bc &= \tfrac{1}{2}\left((a^2+3) + \tfrac{6}{a}\right) \cdot \tfrac{1}{2}\left((a^2+3) - \tfrac{6}{a}\right) \\ &= \tfrac{1}{4}\left((a^2+3)^2 - \tfrac{36}{a^2}\right). \end{aligned}$$

This simplifies to $a^6 + 6a^4 + 29a^2 - 36 = 0$. The cubic formula could be used to find a, but one might observe that $a = 1$ is a solution.

Using $a = 1$, we calculate that $b = \frac{1}{2}\left((a^2+3) + \frac{6}{a}\right) = 5$, and similarly that $c = -1$. Hence our goal is to find the roots of

$$0 = z^4 + 3z^2 + 6z - 5 = (z^2 - z + 5)(z^2 + z - 1).$$

Using the quadratic formula we find that the roots are $z = \frac{1 \pm \sqrt{19}}{2}, \frac{-1 \pm \sqrt{5}}{2}$.

(2) Set up a quartic to find the points of intersection of the parabolas $y = x^2 - 2$ and $x = y^2 - 2$, then find the points of intersection.

We have $x = y^2 - 2 = (x^2 - 2)^2 - 2 = x^4 - 4x^2 - 2$, so the x coordinate of a point of intersection must satisfy the quartic $x^4 - 4x^2 - x - 2 = 0$.

To find the roots of this quartic, we first try to factor

$$x^4 - 4x^2 - x - 2 = (x^2 - ax + b)(x^2 + ax + c)$$

into quadratics. Multiplying the right hand side and equating coefficients we get

$$\begin{aligned} b + c - a^2 &= -4 \\ a(b - c) &= -1 \\ bc &= 2. \end{aligned}$$

The first two of these equations may be rewritten as

$$\begin{aligned} b+c &= -4+a^2 \\ b-c &= -1/a, \end{aligned}$$

or

$$\begin{aligned} b &= \tfrac{1}{2}\left((a^2-4)-\tfrac{1}{a}\right) \\ c &= \tfrac{1}{2}\left((a^2-4)+\tfrac{1}{a}\right). \end{aligned}$$

Thus

$$\begin{aligned} 2 = bc &= \tfrac{1}{2}\left((a^2-4)-\tfrac{1}{a}\right)\cdot\tfrac{1}{2}\left((a^2-4)+\tfrac{1}{a}\right) \\ &= \tfrac{1}{4}\left((a^2-4)^2-\tfrac{1}{a^2}\right). \end{aligned}$$

This simplifies to $a^6-8a^4+8a^2-1=0$. The cubic formula could be used to find a, but one might notice from the symmetry of this equation that $a=1$ is a solution.

Using $a=1$, we calculate that $b=\frac{1}{2}\left((a^2-4)-\frac{1}{a}\right)=-2$, and similarly that $c=-1$. Hence our goal is to find the roots of

$$0=x^4-4x^2-x-2=(x^2-ax+b)(x^2+ax+c)=(x^2-x-2)(x^2+x-1).$$

Factoring the first quadratic and using the quadratic formula to factor the second we obtain

$$0=x^4-4x^2-x-2=[(x-2)(x+1)][(x+\varphi)(x-1/\varphi)],$$

where φ is the Golden Ratio. This means that $x=2,-1,-\varphi$ or $1/\varphi$. The points of intersection are then $(x,y)=(2,2),(-1,-1),(-\varphi,1/\varphi)$, and $(1/\varphi,-\varphi)$.

(3) Solve the system:

$$\begin{aligned} x+y+z &= 4 \\ x^2+y^2+z^2 &= 4 \\ x^3+y^3+z^3 &= 4 \end{aligned}$$

The power sums p_m are related to the elementary symmetric polynomials s_n by the formulas

$$\begin{aligned} p_1 &= s_1, \\ p_2 &= s_1p_1-2s_2, \\ p_3 &= s_1p_2-s_2p_1+3s_3, \end{aligned}$$

ETC. Since we are given that $p_1=p_2=p_3=4$, we derive that $s_1=4, s_2=6, s_3=4$. Hence x,y,z are the roots of $0=t^3-s_1t^2+s_2t-s_3=t^3-4t^2+6t-4$. We depress this with the substitution $t=u+4/3$ and obtain $u^3+\frac{2}{3}u-\frac{20}{27}$. So, $p/3=-2/9$ and $q/2=10/27$ in the Cardano Formula. One root is

$$u=\frac{1}{3}\left(\sqrt[3]{10+\sqrt{108}}+\sqrt[3]{10-\sqrt{108}}\right)=\frac{2}{3},$$

hence $t=u+4/3=2$. The other two values of t can be calculated to be $1+i, 1-i$, so $\{x,y,z\}=\{2,1+i,1-i\}$.

Check:

$$\begin{array}{llll}
2+(1+i)+(1-i) & & =4 & \checkmark \\
2^2+(1+i)^2+(1-i)^2 & =4+(2i)+(-2i) & =4 & \checkmark \\
2^3+(1+i)^3+(1-i)^3 & =8+(-2+2i)+(-2-2i) & =4 & \checkmark
\end{array}$$