HW 5: solution sketches

(1) Use Brahmagupta's method to find a solution to $x^2 - Dy^2 = 1$, where $D = n^2 + 1$.

Choosing $x = n, y = 1$ leads to the starting triple $(x, y, k) = (n, 1, -1)$. Composing it with itself yields a triple $(n^2 + (n^2 + 1)1, 2n, 1) = (2n^2 + 1, 2n, 1)$. Thus $(x, y) = (2n^2 + 1, 2n)$ is a solution.

(2) The quadratic mean of a sequence $a_1, \dots, a_n$ is

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}.$$

Find an integer $n > 1$ such that the quadratic mean of the first n positive integers is again an integer. That is, find $n > 1$ such that

$$\sqrt{\frac{1^2 + 2^2 + \cdots + (n-1)^2 + n^2}{n}}$$

is a positive integer. (Hint: Reduce this problem to Pell's equation using the formula $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.)

If $m = \sqrt{\frac{1^2+2^2+\cdots+n^2}{n}}$, then the hint yields that $m^2 = (n+1)(2n+1)/6$. Multiplying by 48 yields

$$48m^2 = 16n^2 + 24n + 8 = (4n+3)^2 - 1.$$

If $x = 4n + 3$ and $y = 4m$, then this equation is equivalent to

$$x^2 - 3y^2 = 1,$$

which is Pell's equation. We are looking for solutions (x, y) where $x \equiv 3 \pmod 4$ and $y \equiv 0 \pmod 4$.

The continued fraction expansion for $\sqrt{3}$ is

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}},$$

so $\sqrt{3} = [1; \overline{1, 2}]$. Its convergents are $1, 2/1, 5/3, 7/4, \dots$. The first convergent that solves Pell's equation and also satisfies $x \equiv 3 \pmod 4$ and $y \equiv 0 \pmod 4$ is $x = 7$, $y = 4$. In this case, $n = 1$, $m = 1$. But we seek a solution with $n > 1$, so we keep examining convergents:

$$19/11,\ 26/15,\ 71/41,\ 97/56,\ 265/153,\ 362/209,\ 989/571,\ 1351/780.$$

We can stop here, since the denominator $780 = y = 4m$ is divisible by 4 and the numerator $1351 = x = 4n+3$ is congruent to 3 modulo 4. Solving for m and n yields $m = 780/4 = 195$ and $n = (1351-3)/4 = 337$. This means that

$$\sqrt{\frac{1^2+2^2+\cdots+337^2}{337}} = 195.$$

Thus, $n = 337$ is the first $n > 1$ such that the quadratic mean of the first n integers is again an integer. The next example is

$$\sqrt{\frac{1^2+2^2+\cdots+65521^2}{65521}} = 37829.$$

(3) Find the volume formula for a tetrahedron by completing:
  (i) Exercise 4.3.5 from the text.
  (ii) Exercise 4.3.6 from the text.

Let B and H be the base and height of the tetrahedron.

(i) If you dissect the base into 4 congruent triangles, you see that the front prism has two of the triangles as its base while the back prism has one triangle as its base. Hence the area of the base of the front prism is $B/2$ while the area of the back prism is $B/4$. The height of each prism is $H/2$. The back prism has volume $B/4 \cdot H/2 = BH/8$. If you flip the front prism over and join it with the original you get a rectangular prism of volume $B/2 \cdot H/2 = BH/4$, so the original prism has volume $BH/8$. Together the two prisms have volume $BH/8 + BH/8 = BH/4$.

(ii) By 4.3.5, the two prisms together have volume $BH/4$. Call this the "main piece" of the tetrahedron. If the main piece is removed, we are left with two tetrahedra whose linear dimensions are half the dimensions of the original. Each smaller tetrahedron contains each own main piece of volume $((B/4)(H/2))/4 = BH/32$. If you continue this and keep track of the number of main pieces of each mini-tetrahedron you obtain the sum

$$\begin{aligned} & \left(\tfrac{1}{4} + 2\tfrac{1}{4\cdot 8} + 2^2\tfrac{1}{4\cdot 8^2} + \cdots\right)BH \\ = \; & \left(\tfrac{1}{4} + \tfrac{1}{4^2} + \tfrac{1}{4^3} + \cdots\right)BH \\ = \; & \tfrac{1}{3}BH, \end{aligned}$$

where the last equality follows from a calculation we did in class: $\frac{1}{4} + \frac{1}{4^2} + \cdots = \frac{1}{3}$.