

HW 4: solution sketches

- (1) The Euler characteristic of a sphere is 2.
- (a) What is the Euler characteristic of object that is a disjoint union of two spheres?
 - (b) Suppose you move two disjoint spheres closer together until they are tangent. What happens to the Euler characteristic at this moment? Is your answer unexpected? (If so, how do you explain it?)

For (a), triangulate the two spheres. Suppose Sphere 1 has v_1, e_1, f_1 vertices, edges and faces, and Sphere 2 has v_2, e_2, f_2 vertices, edges and faces. The disjoint union therefore has $v_1 + v_2$ vertices, $e_1 + e_2$ edges, and $f_1 + f_2$ faces. Since $v_1 - e_1 + f_1 = 2 = v_2 - e_2 + f_2$, the Euler characteristic of the union is $(v_1 + v_2) - (e_1 + e_2) + (f_1 + f_2) = 2 + 2 = 4$.

For (b), everything works the same way, but we may assume that the point of tangency is a common vertex, so the triangulation of tangent spheres has $v_1 + v_2 - 1$ vertices, $e_1 + e_2$ edges, and $f_1 + f_2$ faces. The Euler characteristic of the tangent spheres is $(v_1 + v_2 - 1) - (e_1 + e_2) + (f_1 + f_2) = 2 + 2 - 1 = 3$.

The answer may seem odd, since in the classification of compact, connected, orientable, 2-dimensional surfaces all Euler characteristics are even. But here, the Euler characteristic is odd. The tangent spheres are compact, connected, orientable and 2-dimensional. But tangent spheres are not a surface, according to the standard definition. On a surface, every point has a small neighborhood that looks like a piece of the plane. The point of tangency has no such neighborhood.

- (2) Find the regular continued fraction expansion of \sqrt{p} for the values $p = 7, 11, 13$.

Using a calculator, we can start producing some values. Once it appears to become periodic, we can guess the full expansion and prove its correctness.

(b) $p = 11$.

$$\sqrt{11} = 3 + \frac{1}{3 + \frac{1}{6 + \frac{1}{3 + \frac{1}{6 + \dots}}}}$$

so $\sqrt{11} = [3; \overline{3, 6}]$.

Check: Let $x = [0; \overline{3, 6}]$. Then

$$x = 0 + \frac{1}{3 + \frac{1}{6 + x}}$$

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This holds exactly when $x = \frac{6+x}{19+3x}$, or $3x^2 + 19x = 6 + x$, or $3x^2 + 18x - 6 = 0$, or $x^2 + 6x - 2 = 0$. According to the quadratic formula, this means $x = \frac{-6 \pm \sqrt{44}}{2} = -3 + \sqrt{11}$. Since $-3 + \sqrt{11} = x = [0; \overline{3, 6}]$, we get $\sqrt{11} = [3; \overline{3, 6}]$.

(a) $p = 7$.

$$\sqrt{7} = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \dots}}}}}}}},$$

so $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$.

(c) $p = 13$.

$$\sqrt{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \dots}}}}}},$$

so $\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$.

(3) Find an integer solution to $x^2 - py^2 = 1$. for each $p = 7, 11, 13$.

(b) $p = 11$.

The convergents are $[3], [3; 3], [3; 3, 6], [3; 3, 6, 3]$, ETC. These are $3/1, 10/3, 63/19, 199/60$, ETC. Testing these yields a first solution (x, y) when $x/y = 10/3$, or $(x, y) = (10, 3)$.

(a) $p = 7$.

The convergents are $2/1, 3/1, 5/2, 8/3$, ETC. Testing these yields a first solution (x, y) when $x/y = 8/3$, or $(x, y) = (8, 3)$.

(c) $p = 13$.

The convergents are $3/1, 4/1, 7/2, 11/3$, ETC. Testing these yields a first solution (x, y) when $x/y = 649/180$, or $(x, y) = (649, 180)$.