HW 3: solution sketches

(1) It is not possible to construct an angle of $\pi/13$ radians with straightedge and compass. Show that it is nevertheless possible to trisect an angle of $\pi/13$ with straightedge and compass. (That is, if you are given an angle of $\pi/13$, then from it you can construct an angle of $\pi/39$.)

Suppose that you are given an angle of $\pi/13$. Construct an angle of $\pi/3$, and form integer combinations $m(\pi/3) + n(\pi/13)$ by copying the orginal angles repeatedly in one direction or the other. One of these combinations (when $m = 1$ and $n = -4$) is $1(\pi/13) - 4(\pi/13) = \pi/39$. This is a method of constructing $\pi/39$ from $\pi/13$.

(2) Show that if a polygon is constructible, then its area is a constructible number. (Hint: start with triangles.)

It is enough to solve the problem for triangles. For, if P is a constructed polygon, its vertices are constructible points. Connect some of these by segments to decompose the P into constructible triangles $T_1 \cup \cdots \cup T_n$. Each T_i is a constructible triangle, so if we have solved this problem for triangles, then $\mathrm{Area}(T_i)$ is a constructible number. Therefore, $\mathrm{Area}(P) = \sum \mathrm{Area}(T_i)$ is also constructible.

We have reduced the problem to triangles. Suppose that T is a constructible triangle. Copy it along the x-axis so that one vertex is at the origin and another vertex lies on the positive side of the axis. The base of the triangle is constructible, since it is a side length of a constructible triangle. I claim that the height is also constructible. To see this, it is enough to drop a perpendicular line to the x-axis from the vertex that is not on the x-axis. (Dropping perpendiculars is possible with straightedge and compass.) Since this construction is possible, the height of the triangle is a constructible length.

Since the base and height of T are constructible numbers,

$$\mathrm{Area}(T_i) = \frac{1}{2}\text{base} \times \text{height}$$

is a constructible number.

(3) Show that if a regular polygon of circumradius 1 has constructible area, then it is possible to construct a copy of the polygon. (The circumradius is the radius of the circumscribing circle.)

Stage 1: First show that the area of a regular n-gon with circumradius 1 is $n \sin(\pi/n) \cos(\pi/n)$. (For this, divide the regular n-gon into n triangles, compute the area of each one, then add the results.)

Stage 2: Observe that $n \sin(\pi/n) \cos(\pi/n) = (n/2) \sin(2\pi/n)$. Since $n/2$ is rational, $(n/2) \sin(2\pi/n)$ is a constructible number iff $\sin(2\pi/n)$ is a constructible number iff $\cos(2\pi/n)$ is a constructible number iff a regular n-gon is constructible.