HW 2: solution sketches

(1) What is the height of a regular tetrahedron of side length 1?

There are various ways to do this, but the simplest might be to find a nice regular tetrahedron in space (not necessarily of side length 1), and then compute the ratio of the height to the side length.

There is a nice regular tetrahedron in 4-space, namely the one with vertices at $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$. If the first three define the bottom face, then its center is at $(e_1 + e_2 + e_3)/3 = (1/3, 1/3, 1/3, 0)$. The distance from the top, e_4, to this point is $\|e_4 - (e_1 + e_2 + e_3)/3\|$, or

$$\sqrt{(1/3)^2 + (1/3)^2 + (1/3)^2 + 1} = 2\sqrt{3}/3.$$

The side length of this tetrahedron is the length from e_1 to e_2, which is $\|e_1 - e_2\| = \sqrt{2}$. The desired ratio is now $(2\sqrt{3}/3)/\sqrt{2}) = \sqrt{6}/3 = \sqrt{2/3}$.

(2) Exercise 2.2.2 from the text.

We can choose coordinates so that the center of the figure on page 22 is at $(0, 0, 0)$, the x-axis lies on the horizontal rectangle so that a particle traveling in the positive x-direction passes midway between points B and C, and the z axis is the vertical direction. With these choices (and the fact that each rectangle is $1 \times \varphi$) we find that the coordinates of A, B, C are $(1/2, 0, \varphi/2)$, $(\varphi/2, -1/2, 0)$, $(\varphi/2, 1/2, 0)$.

Now, the length BC is 1. The length AB is

$$\sqrt{(1/2 - \varphi/2)^2 + (1/2)^2 + (\varphi/2)^2} = \sqrt{1/2 - \varphi/2 + \varphi^2/2} = \sqrt{1/2 + 1/2} = 1.$$

Similarly AC has length 1. Thus, a typical face $\triangle ABC$ is equilateral, as desired.

(3) Let P be a polyhedron. Suppose $F_1, \ldots, F_k$ are the faces of P that meet at vertex V, and that $A_1, \ldots, A_k$ are the angles of these faces at V. Define the defect at vertex V to be (360-(sum of the angles A_i)). (For example, in a cube there are three squares meeting at any vertex, so the defect at any vertex is $(360-(90 + 90 + 90)) = 90$ degrees.) The total defect of P is the sum of the defects at all of the vertices of P. Exercise: find the total defect of each of the Platonic solids.

(a) (Tetrahedron) Vertex defect $= 360 - 3(60) = 180$. Total defect $= 4(180) = 720°$.
(b) (Cube) Vertex defect $= 360 - 3(90) = 90$. Total defect $= 8(90) = 720°$.
(c) (Octahedron) Vertex defect $= 360 - 4(60) = 120$. Total defect $= 6(140) = 720°$.
(d) (Dodecahedron) Vertex defect $= 360 - 3(108) = 36$. Total defect $= 20(36) = 720°$.
(e) (Icosahedron) Vertex defect $= 360 - 5(60) = 60$. Total defect $= 12(60) = 720°$.